J. Math. Fluid Mech. 14 (2012), 501–528

1422-6928/12/030501-28
DOI 10.1007/s00021-011-0077-7

Journal of Mathematical
Fluid Mechanics

Generalized Flows Satisfying Spatial Boundary Conditions

B. Buffoni

Communicated by T. Beale

Abstract. In a region D in $\mathbb{R}^2$ or $\mathbb{R}^3$, the classical Euler equation for the regular motion of an inviscid and incompressible fluid of constant density is given by

$$\partial_t v + (v \cdot \nabla_x)v = -\nabla_x p,\ \mathrm{div}_x v = 0,$$

where $v(t,x)$ is the velocity of the particle located at $x \in D$ at time t and $p(t,x) \in \mathbb{R}$ is the pressure. Solutions v and p to the Euler equation can be obtained by solving

$$\begin{cases} \nabla_x \left\{ \partial_t \phi(t,x,a) + p(t,x) + (1/2)|\nabla_x \phi(t,x,a)|^2 \right\} = 0 \text{ at } a = \kappa(t,x), \\ v(t,x) = \nabla_x \phi(t,x,a) \text{ at } a = \kappa(t,x), \\ \partial_t \kappa(t,x) + (v \cdot \nabla_x)\kappa(t,x) = 0, \\ \mathrm{div}_x v(t,x) = 0, \end{cases} \tag{0.1}$$

where

$$\phi : \mathbb{R} \times D \times \mathbb{R}^l \to \mathbb{R} \text{ and } \kappa : \mathbb{R} \times D \to \mathbb{R}^l$$

are additional unknown mappings ($l \geq 1$ is prescribed). The third equation in the system says that $\kappa \in \mathbb{R}^l$ is convected by the flow and the second one that ϕ can be interpreted as some kind of velocity potential. However vorticity is not precluded thanks to the dependence on a. With the additional condition $\kappa(0,x) = x$ on D (and thus $l = 2$ or 3), this formulation was developed by Brenier (Commun Pure Appl Math 52:411–452, 1999) in his Eulerian–Lagrangian variational approach to the Euler equation. He considered generalized flows that do not cross ∂D and that carry each "particle" at time $t = 0$ at a prescribed location at time $t = T > 0$, that is, $\kappa(T,x)$ is prescribed in D for all $x \in D$. We are concerned with flows that are periodic in time and with prescribed flux through each point of the boundary ∂D of the bounded region D (a two- or three-dimensional straight pipe). More precisely, the boundary condition is on the flux through ∂D of particles labelled by each value of κ at each point of ∂D. One of the main novelties is the introduction of a prescribed "generalized" Bernoulli's function $H : \mathbb{R}^l \to \mathbb{R}$, namely, we add to (0.1) the requirement that

$$\partial_t \phi(t,x,a) + p(t,x) + (1/2)|\nabla_x \phi(t,x,a)|^2 = H(a) \text{ at } a = \kappa(t,x) \tag{0.2}$$

with ϕ, p, κ periodic in time of prescribed period $T > 0$. Equations (0.1) and (0.2) have a geometrical interpretation that is related to the notions of "Lamb's surfaces" and "isotropic manifolds" in symplectic geometry. They may lead to flows with vorticity. An important advantage of Brenier's formulation and its present adaptation consists in the fact that, under natural hypotheses, a solution in some weak sense always exists (if the boundary conditions are not contradictory). It is found by considering the functional

$$(\kappa, v) \to \int_0^T \int_D \left\{ \frac{1}{2}|v(t,x)|^2 + H(\kappa(t,x)) \right\} dt\, dx$$

defined for κ and v that are T-periodic in t, such that

$$\partial_t \kappa(t,x) + (v \cdot \nabla_x)\kappa(t,x) = 0,\ \mathrm{div}_x v(t,x) = 0,$$

and such that they satisfy the boundary conditions. The domain of this functional is enlarged to some set of vector measures and then a minimizer can be obtained. For stationary planar flows, the approach is compared with the following standard minimization method: to minimize

$$\int_{]0,L[\times]0,1[} \{(1/2)|\nabla \psi|^2 + H(\psi)\} dx \text{ for } \psi \in W^{1,2}(]0,L[\times]0,1[)$$

Birkhäuser

under appropriate boundary conditions, where ψ is the stream function. For a minimizer, corresponding functions ϕ and κ are given in terms of the stream function ψ.

Mathematics Subject Classification (2010). Primary 35A15; Secondary 35D05.

Keywords. Generalized flows, Boundary conditions, Primal and dual problems, Variational methods.

Contents

1. Introduction

We first explain the formulation with usual functions, and then generalize it to vector measures, so that the minimization problem we shall consider has a minimizer in the more general setting.

Let the section $\Sigma \subset \mathbb{R}^{d-1}$ be the closure of a bounded, open, connected and non-empty set, where $d \in \{2, 3\}$ is the spatial dimension. Let D be the region

$$D = [0, L] \times \Sigma,\ x = (y, z) \in D \text{ with } y \in [0, L] \text{ and } z \in \Sigma,$$

where $L > 0$ is its length.

The classical Euler equation for the regular motion of an inviscid and incompressible fluid of constant density in the region D can be written

$$\partial_t v + (v \cdot \nabla_x)v = -\nabla_x p,\ \mathrm{div}_x v = 0, \tag{1.1}$$

where $v(t, x) \in \mathbb{R}^d$ is the velocity of the particle located at $x \in D$ at time t and $p(t, x) \in \mathbb{R}$ is the pressure at $x \in D$ at time t. We are concerned with flows that are periodic in time of prescribed period $T > 0$, and that have prescribed flux through each point of the boundary ∂D of the bounded region D.

Solutions v and p to the Euler equation can be obtained by solving (0.1), where

$$\phi : \mathbb{R} \times D \times \mathbb{R}^l \to \mathbb{R} \text{ and } \kappa : \mathbb{R} \times D \to \mathbb{R}^l$$

are additional unknown mappings that are time periodic of period T ($l \geq 1$ is prescribed).

If the flow is stationary (v and p do not depend on time), the Bernoulli equation states that $\frac{1}{2}|v|^2 + p := \widetilde{H}$ is constant along stream lines, that is, each particle is associated with a particular value of $\widetilde{H}$. In this case, $\widetilde{H}$ is called "the Bernoulli function" (or "hydraulic head", "Bernoulli's head", "total pressure", etc).

Given $H : \mathbb{R}^l \to \mathbb{R}$, we add to (0.1) the requirement (0.2). Observe that $-tH(a)$ cannot be added to ϕ in general because of the periodicity condition on t. The function H amounts to $\widetilde{H}$ when $\partial_t \phi = 0$ everywhere and therefore we shall call H the "generalized Bernoulli function".

Concerning the boundary conditions on ∂D, we denote by n the outward unitary normal vector to ∂D (almost everywhere well defined on ∂D),

$$\mu_0(t, x) := -v(t, x) \cdot n(x) \text{ for } x \in \{0\} \times \mathrm{int}\,\Sigma$$

and

$$\mu_L(t,x) := v(t,x)\cdot n(x) \text{ for } x \in \{L\}\times \text{int}\,\Sigma.$$

We shall require that

1. $v(t,x)\cdot n(x) = 0$ for all $x \in]0,L[\times\partial\Sigma$
   (the flow does not cross $]0,L[\times\partial\Sigma$),
2. $\mu_0(t,x) \geq 0$ and $\mu_L(t,x) \geq 0$ are prescribed on $\{0\}\times \text{int}\,\Sigma$ and $\{L\}\times \text{int}\,\Sigma$,
3. and $\kappa(t,x)$ is prescribed on $\{0\}\times \text{int}\,\Sigma$ and $\{L\}\times \text{int}\,\Sigma$
   (that is, the ingoing flux through $\{0\}\times \text{int}\,\Sigma$, the outgoing flux through $\{L\}\times \text{int}\,\Sigma$ and κ are prescribed).

To understand the geometrical meaning of (0.1) and (0.2) when p, ϕ, κ and H are smooth, observe that, given any smooth path

$$[0,1] \ni s \to \gamma(s) = (\gamma_t(s), \gamma_x(s)) \in \mathbb{R}\times \text{int}\,D$$

such that $\gamma(0) = \gamma(1)$, the following integral

$$\int_0^1 \left\{ \left(-p(\gamma(s)) - (1/2)|v(\gamma(s))|^2\right)\gamma_t'(s) + v(\gamma(s))\cdot\gamma_x'(s)\right\} ds \tag{1.2}$$

is invariant if we let the path evolve in $\mathbb{R}\times \text{int}\,D$ accordingly to the flow defined by the velocity field $(1,v) \in \mathbb{R}\times\mathbb{R}^d$, as long as $\mathbb{R}\times\partial D$ is not reached (the first component is the time variable). Indeed, this flow is governed by the Hamiltonian

$$\mathbb{R}\times \text{int}\ D\times\mathbb{R}\times\mathbb{R}^d \ni (t,x,h,v) \to h + p(t,x) + (1/2)|v|^2, \tag{1.3}$$

where h is the variable conjugate to time, the convective derivative of which is equal to $-\partial_t p(t,x)$. The preservation of (1.2) is a consequence of the preservation of the Hamiltonian (1.3), which can therefore be chosen to vanish everywhere, and of the integral

$$\int_0^1 \{\, h(\gamma(s))\gamma_t'(s) + v(\gamma(s))\cdot\gamma_x'(s)\,\}\, ds.$$

The preservation of the latter integral is a consequence of the preservation of the standard symplectic 2-form on $(\mathbb{R}\times\mathbb{R}^d)\times(\mathbb{R}\times\mathbb{R}^d)$. When γ_t is constant, this amounts to the well known fact that the circulation along a "material" loop is preserved. Let us now consider the case where, for each a, the subset of $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$

$$W_a = \Big\{ \left(t,x,-p(t,x) - (1/2)|v(t,x)|^2, v(t,x)\right) : \\ (t,x)\in\mathbb{R}\times \text{int}\,D,\ \kappa(t,x) = a\Big\}$$

is either empty or a smooth manifold. Note that W_a is globally invariant by the flow (the components corresponding to a fluid particle stay in W_a as long as they remain in $\mathbb{R}\times \text{int}\,D\times\mathbb{R}^{1+d}$; indeed κ is convected by the flow). If (0.1) and (0.2) are satisfied, for all smooth paths

$$[0,1]\ni s\to\gamma(s) = (\gamma_t(s),\gamma_x(s)) \in \kappa^{-1}(\{a\})\cap(\mathbb{R}\times \text{int}\,D) \subset \mathbb{R}\times\mathbb{R}^d$$

such that $\gamma(0) = \gamma(1)$ and that are smoothly contractible in $\kappa^{-1}(\{a\})\cap(\mathbb{R}\times \text{int}\,D)$, it holds

$$\int_0^1 \left\{-\left(p(\gamma(s)) + (1/2)|v(\gamma(s))|^2\right)\gamma_t'(s) + v(\gamma(s))\cdot\gamma_x'(s)\right\} ds = 0$$

because

$$\begin{aligned}
&\int_0^1 \left\{ \left(-p(\gamma(s)) - (1/2)|v(\gamma(s))|^2\right) \gamma_t'(s) + v(\gamma(s)) \cdot \gamma_x'(s) \right\} ds \\
&\quad = \int_0^1 \left\{ \left(H(a) - p(\gamma(s)) - (1/2)|v(\gamma(s))|^2\right) \gamma_t'(s) + v(\gamma(s)) \cdot \gamma_x'(s) \right\} ds \\
&\quad = \int_0^1 \left\{ \partial_t \phi(\gamma(s), a)\gamma_t'(s) + \nabla_x \phi(\gamma(s), a) \cdot \gamma_x'(s) \right\} ds \\
&\quad = \int_0^1 \frac{d}{ds}\phi(\gamma(s), a) ds = \phi(\gamma(1), a) - \phi(\gamma(0), a) = 0.
\end{aligned}$$

The manifold W_a is therefore said to be *isotropic* (equivalently, the standard symplectic two-form on $\mathbb{R}^{1+d} \times \mathbb{R}^{1+d}$ vanishes when restricted to each of its tangent spaces; see e.g. [1]).[1] In fact this analysis still holds if the paths are not necessarily contractible and we say therefore that this manifold is isotropic and *exact*. On the other hand, consider a smooth path

$$\begin{aligned}
&\gamma = (\gamma_t, \gamma_x) : [0,1] \to \kappa^{-1}(\{a\}) \cap (\mathbb{R} \times \operatorname{int} D) \subset \mathbb{R} \times \mathbb{R}^d \\
&\qquad\qquad \text{such that } \gamma_t(0) = 0,\ \gamma_t(1) = T \text{ and } \gamma_x(1) = \gamma_x(0)
\end{aligned} \tag{1.4}$$

where $T > 0$ is the period. We now get

$$\int_0^1 \left\{ -\left(p(\gamma(s)) + (1/2)|v(\gamma(s))|^2\right) \gamma_t'(s) + v(\gamma(s)) \cdot \gamma_x'(s) \right\} ds = -TH(a)$$

because

$$\begin{aligned}
&\int_0^1 \left\{ \left(-p(\gamma(s)) - (1/2)|v(\gamma(s))|^2\right) \gamma_t'(s) + v(\gamma(s)) \cdot \gamma_x'(s) \right\} ds = -TH(a) \\
&\quad + \int_0^1 \left\{ \left(H(a) - p(\gamma(s)) - (1/2)|v(\gamma(s))|^2\right) \gamma_t'(s) + v(\gamma(s)) \cdot \gamma_x'(s) \right\} ds \\
&= -TH(a).
\end{aligned}$$

If $H(a) \neq 0$, this means that, seen as a subset of $((\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^d) \times \mathbb{R}^{1+d}$, W_a is not exact, and thus $H(a)$ is a measure of non-exactness in $((\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^d) \times \mathbb{R}^{1+d}$ (if such a path γ exists). Here $\mathbb{R}/T\mathbb{Z}$ is regarded as a compact one-dimensional manifold (a "circle"). If κ has no critical point on $\mathbb{R} \times D$, then $l \leq 1 + d$ and the dimension of W_a (if not empty) is $1 + d - l$.

Hence if $l \in \{1, \ldots, d-1\}$ and if equations (0.1) and (0.2) can be solved with smooth p, ϕ, κ and κ without critical points on $\mathbb{R} \times D$, then, given any a such that

$$\{ (t, x) \in \mathbb{R} \times \{0\} \times \operatorname{int} \Sigma :\ \kappa(t, x) = a \}$$

and

$$\{ (t, x) \in \mathbb{R} \times \{L\} \times \operatorname{int} \Sigma :\ \kappa(t, x) = a \}$$

[1] As W_a is invariant, W_a is isotropic exactly when $\{(x, v(t,x)) : x \in \operatorname{int} D\}$ is isotropic in $\mathbb{R}^d \times \mathbb{R}^d$ for all $t \in \mathbb{R}$. Indeed, for every $(t,x) \in \mathbb{R} \times \operatorname{int} D$ such that $\kappa(t,x) = a$, the vector $(1, v(t,x), -\partial_t p(t,x), -\nabla_x p(t,x))$ is tangent to W_a at $(t, x, -p(t,x) - (1/2)|v(t,x)|^2, v(t,x)) \in W_a$. Moreover the standard symplectic form in $\mathbb{R}^{1+d} \times \mathbb{R}^{1+d}$ vanishes at the pair of tangent vectors to W_a given by $(1, v(t,x), -\partial_t p(t,x), -\nabla_x p(t,x))$ and $(0, \delta x, -\nabla_x p(t,x) \cdot \delta x - v(t,x) \cdot (\partial_x v(t,x)\delta x), \partial_x v(t,x)\delta x)$ for all appropriate $\delta x \in \mathbb{R}^d$.

are smooth manifolds of dimensions $\le d-l$, the above manifold W_a in $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$ is of dimension at least 2, isotropic, exact and

$$H(a)=\frac{1}{T}\int_0^1\left\{\,\left(p(\gamma(s))+(1/2)|v(\gamma(s))|^2\,\right)\gamma_t'(s)-v(\gamma(s))\cdot\gamma_x'(s)\,\right\}ds$$

for all smooth paths γ satisfying (1.4).

To sum up, in this context, the problem at hand can be approximately interpreted as the one of "extending" manifolds defined in $\mathbb{R}\times\{0,L\}\times\operatorname{int}\Sigma$ to invariant exact isotropic manifolds in $\mathbb{R}\times\operatorname{int}D\times\mathbb{R}^{1+d}$, in a way compatible with the above interpretation of $H(a)$, with the boundary conditions on the flux and with the Euler equation (in particular, in a compatible way with the existence of a pressure field p and the equation $\operatorname{div}_x v=0$). In the autonomous three-dimensional case, this is related to the problem of extending Lamb's submanifolds from the boundary to the full domain (see Sect. 3).

Given the continuous generalized Bernoulli function $H:\mathbb{R}^l\to\mathbb{R}$, we consider the functional

$$(\kappa,v)\to\int_0^T\int_D\left\{\frac{1}{2}|v(t,x)|^2+H(\kappa(t,x))\right\}dt\,dx$$

defined for κ and v that are T-periodic in t, such that

$$\partial_t\kappa(t,x)+(v\cdot\nabla_x)\kappa(t,x)=0,\ \operatorname{div}_x v(t,x)=0,$$

and such that they satisfy the above boundary conditions.

The first step is to enlarge the class of admissible functions κ and v, so that a minimizer can be found (the value of the infimum remains the same or decreases). Instead of classical flows, we shall look for, and find, time-periodic "generalized flows" [8,18]. In a generalized flow, a particle at a given time is no more necessary located at a point, but should rather be considered as a probability measure. Moreover the supports of the measures corresponding to two different "particles" are not necessarily disjoint. We denote by $T>0$ the given time period and we replace the set of values of κ in $\mathbb{R}^l$ by a "space of labels" A that we assume to be a compact metric space. It consists of Lagrangian labels allowing to distinguish (more ore less finely) the "particles"; the variable $a\in A$ has a Lagrangian nature; for example a can represent the location and time of the "particle" when it crosses the section $y=0$ if indeed each particle crosses this section exactly once and at a precise point. The generalized Bernoulli function $a\to H(a)$ is defined on A and will be given a priori.

We shall use the notations

$$Q=(\mathbb{R}/T\mathbb{Z})\times D\text{ and }Q'=Q\times A=(\mathbb{R}/T\mathbb{Z})\times D\times A,$$

where $t\in\mathbb{R}/T\mathbb{Z}$ denotes time defined up to a multiple of the period T.

More precisely, by "generalized flow", we mean a pair (c,m) such that c is a Borelian measure on Q' with values in $[0,\infty)$, and m is a vector Borelian measure on Q' with values in $\mathbb{R}^d$ (that is, each component of m is a finite signed Borelian measure on Q').

For such generalized flow $(c,m),m$ may have a density $v:Q'\to\mathbb{R}^d$ with respect to c, that is,

$$m(E)=\int_E v\,dc,\ E\subset Q'$$

(E being restricted to be a Borelian subset). In this case, $(t,x,a)\to v$ can be interpreted as a generalized vector field (generalized in the sense that it may depend on a too). If the generalized flow (c,m) corresponds to a classical flow, then such a density v exists and there is a map $\kappa:Q\to A$ such that the flow is carried by the graph of κ:

$$c(\{(t,x,a)\in Q':a\neq\kappa(t,x)\})=0.$$

In other words, the label of a particle at (x, t) is uniquely defined and, as there is no more way of distinguishing various particles at (t, x), we can consider that there is only one particle at almost all (t, x).

In order that (c, m) describes an incompressible flow with constant density, (c, m) has to satisfy other conditions that we shall explain later. Moreover we would like the flow to behave in a given way at the spatial boundary ∂D, but, as we now work with measures, the above functions μ_0 and μ_L can be replaced by measures too. More specifically, we shall require that the flow does not cross $[0, L] \times \partial\Sigma$ and that its ingoing flux through $\{0\} \times \Sigma$ and its outgoing flux through $\{L\} \times \Sigma$ are given a priori. For a generalized flow (c, m), the ingoing flux and the outgoing flux are described by measures

$$\mu_0 \text{ and } \mu_L \quad \text{on} \quad (\mathbb{R}/T\mathbb{Z}) \times \{0\} \times \Sigma \times A \quad \text{and} \quad (\mathbb{R}/T\mathbb{Z}) \times \{L\} \times \Sigma \times A \quad \text{(resp.)}$$

with values in $[0, \infty)$.

Given the continuous generalized Bernoulli function $H : A \to \mathbb{R}$, we shall find generalized solutions to the Euler equation (1.1) by minimizing

$$\int_{Q'} \{(1/2)|v|^2 + H\} dc \tag{1.5}$$

over an appropriate convex set of generalized flows (c, m), where we consider the integral as equal to $+\infty$ at (c, m) if m does not have a density v with respect to c. As this functional is convex in (c, m), the existence of a minimizer is obtained by standard arguments if it is finite at least at one (c, m).

In Sect. 2, we shall develop more fully this variational approach by defining the functional space of measures and by explaining a dual formulation analogous to the one by Brenier [8]. The dual problem is the maximization over an appropriate class of admissible functions $\phi : Q' \to \mathbb{R}$ and $p : Q \to \mathbb{R}$ of the functional

$$\int_{(\mathbb{R}/T\mathbb{Z}) \times \{L\} \times \Sigma \times A} \phi(\cdot, L, \cdot, \cdot) d\mu_L - \int_{(\mathbb{R}/T\mathbb{Z}) \times \{0\} \times \Sigma \times A} \phi(\cdot, 0, \cdot, \cdot) d\mu_0 + \int_Q p(t, x) dt dx$$

under the constraint

$$\partial_t \phi + p + (1/2)|\nabla_x \phi|^2 \leq H \text{ everywhere on } Q'. \tag{1.6}$$

Under the hypothesis that there is indeed a maximizer (this is a delicate issue), a minimizer (c, m) of the "primal" problem and a maximizer of the dual problem are related by

$$\partial_t \phi + p + (1/2)|\nabla_x \phi|^2 = H \text{ and } v = \nabla_x \phi \quad c\text{-almost everywhere.} \tag{1.7}$$

The gradient is with respect to the spatial variable x and ϕ can depend on a. In this relationship, found in his setting by Brenier, there is a kind of velocity potential ϕ, but flows with vorticity can still be described because of the dependence on a. The novelty here is the introduction of H and the fact that initial and final conditions on the generalized flows in [8] are replaced by spatial boundary conditions and periodicity in time. At the end of Sect. 2, we briefly recall Brenier's setting.

A difficult issue is to check whether a minimizer of (1.5) corresponds indeed to a classical solution. Our aim is more modest: firstly, we would like to get convinced that (1.7) is a generalization of Euler's equation (1.1), and a natural way of doing it is to show that (1.7) implies (1.1) for a minimizer that is not merely a generalized flow but that corresponds to a classical velocity vector field (see Sect. 3). In passing, we explain in Proposition 3.2 that, for such a classical minimizer, the path of each particle is a minimizer of a related one-dimensional variational integral. Secondly, we would like to show that some well known two-dimensional classical solutions seen as generalized flows do indeed minimize (1.5) (see Sects. 4 and 5). Moreover, for these examples, we would like to give a corresponding function ϕ explicitly in terms of the stream function. We focus on two-dimensional steady flows ($d = 2$), but it should be possible to get analogous results for particular classes of three-dimensional flows with additional symmetries (so that the problem is essentially two-dimensional).

In Sect. 4, we consider

$$d = 2,\ \Sigma = [0,1],\ x = (y,z) \in D = [0,L] \times \Sigma,\ A = [0,1]$$

and $H : A \to \mathbb{R}$ continuous. We then compare the minimization of

$$\int_{(\mathbb{R}/T\mathbb{Z})\times D\times A} \{(1/2)|v|^2 + H(a)\} dc \tag{1.8}$$

among all generalized flows (c,m) that satisfy appropriate spatial boundary conditions, with the minimization of

$$\int_0^L \int_0^1 \{(1/2)|\nabla\psi|^2 + H(\psi)\} dy\, dz \tag{1.9}$$

over all functions $\psi : D \to \mathbb{R}$ that are regular enough and that satisfy the boundary condition $\psi = \psi_0$ on ∂D, where $\psi_0 : D \to \mathbb{R}$ is such that

$$\psi_0(y,0) = 0 \ \text{ and } \psi_0(y,1) = 1 \text{ for all } y \in [0,L].$$

The latter problem is classical and used in many applications (see for example [5]). Under quite general assumptions, a minimizer ψ exists and can be interpreted as the stream function of a steady flow satisfying the Euler equation (1.1). Seen as a generalized flow, we show that it is a minimizer of (1.8) and give a corresponding ϕ, which satisfies (1.6) and (1.7) (for a precise statement, see Proposition 4.2 and the remark that follows it).

However we rely on the technical hypothesis (4.6) below. We replace it by a standard convexity hypothesis on H in Sect. 5 by minimizing (1.8) over a smaller class of generalized flows, namely flows (c,m) that satisfy the additional condition

$$\int_A ac(t,y,z,da) = \int_0^z \left\{ \int_A v_y(t,y,s,a) c(t,y,s,da) \right\} ds \tag{1.10}$$

for almost all (t,y,z), where v_y denotes the first component of the two-dimensional velocity field $v = (v_y, v_z)$. The comparison between (1.8) and (1.9) becomes simpler. Moreover (1.10) is a linear condition on (c,m) that has a natural interpretation when (c,m) corresponds to a classical flow: for a particle located at (t,y,z), the value of a is the flux at time t through the section $\{y\} \times [0,z]$. This is to be compared with the meaning of the steady stream function ψ in (1.9): $\psi(y,z)$ is the flux through the section $\{y\} \times [0,z]$. We believe that conditions similar to (1.10) may be helpful in three-dimensional problems with symmetries. Equation (1.7) becomes

$$\partial_t\phi + p + a\partial_z G + \frac{1}{2}|v|^2 = H(a) \text{ and } v = (\partial_y\phi + G, \partial_z\phi) \ c\text{-almost everywhere} \tag{1.11}$$

for some continuous functions $\phi : (t,x,a) \to \mathbb{R}$ and $G : x = (y,z) \to \mathbb{R}$ with continuous derivatives $\nabla_x\phi$ and $\partial_z G$, and such that $G(y,1) = 0$ for all $y \in [0,L]$. However, in order that (1.11) can be seen as a generalization of (1.1), it is convenient to restrict (1.11) to steady flows (see the hypotheses in Proposition 5.2). The advantage of (1.11) over (1.7) is that, in (1.7), we cannot expect ϕ to be a-independent if there is vorticity, whereas in (1.11) ϕ can be a-independent. In Proposition 5.3, for the classical minimization problem (1.9), we explicitly give the new corresponding function ϕ, which indeed does not depend on a anymore. When the vorticity is constant, such a ϕ is used and called "generalized velocity potential" in works on Hamiltonian formalism for surface waves with constant vorticity [10,20].

In Sect. 6, we extend (1.9) to a classical variational approach for three-dimensional stationary classical flows. The stream function ψ is replaced by a pair of functions f and g, and the functional (1.9) is replaced by

$$\int_D \{(1/2)|\nabla f \times \nabla g|^2 + H(f,g)\}dx, \tag{1.12}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $D \subset \mathbb{R}^3$. The minimization problem (1.5) for three-dimensional stationary generalized flows can be then considered as a relaxation of (1.12), but the existence of a minimizer of (1.5) is easy to ensure among generalized flows.

In the Conclusion, we briefly comment on the relationship with Mather's measures.

2. Duality

We adapt Brenier's theory to spatial boundary conditions and observe that rotational flows could emerge in this way. We refer to [4] for an introduction to variational methods in hydrodynamics, and in particular to Sections II.2. and IV.7.

Remember the notations

$$d \in \{2,3\},\ D = [0,L] \times \Sigma,\ x = (y,z) \in D \text{ with } y \in [0,L] \text{ and } z \in \Sigma,$$
$$Q = (\mathbb{R}/T\mathbb{Z}) \times D \text{ and } Q' = Q \times A = (\mathbb{R}/T\mathbb{Z}) \times D \times A,$$

where we assume that A is a compact metric space.

Let μ_0 and μ_L be two Borelian measures that are non-negative and finite on $(\mathbb{R}/T\mathbb{Z}) \times \Sigma \times A$. Also let $H : A \to \mathbb{R}$ be continuous.

Admissible functions. The function $\phi : Q' \to \mathbb{R}$ is said *admissible* if it is continuous, differentiable with respect to the variables

$$(t,x) \in (\mathbb{R}/T\mathbb{Z}) \times \operatorname{int} D,$$

its derivatives having continuous extensions to Q'. The function $p : Q \to \mathbb{R}$ is *admissible* if $p \in C(Q)$ (without necessarily having a zero spatial mean). Note that functions in $C(Q)$ and $C(Q')$ are periodic in t, as $\mathbb{R}/T\mathbb{Z}$ is used in the definition of Q and Q'. We shall use the notations ∇_x and ∂_x without distinguishing them, and also ∇_z and ∂_z.

Generalized flows. We shall call "generalized flow" a pair (c,m) such that c is a Borelian measure that is finite and non negative on Q', m is a vector Borelian measure on Q' with values in $\mathbb{R}^d$, and such that

$$\int_{Q'} (\partial_t \phi + p)dc + \int_{Q'} \nabla_x \phi \cdot dm = \int_{(\mathbb{R}/T\mathbb{Z})\times D} p(t,x)dtdx$$

for all admissible ϕ whose supports are included in $(\mathbb{R}/T\mathbb{Z})\times]0,L[\times\Sigma \times A$ and for all admissible p. This implies that the push-forward of c by the projection $(t,x,a) \to (t,x)$ is the Lebesgue measure on $(\mathbb{R}/T\mathbb{Z}) \times D$.

Primal problem. We seek a finite Borelian measure c on Q' and a vector Borelian measure m on Q' with values in $\mathbb{R}^d$ that realize the following infimum:

$$\inf \int_{Q'} \{(1/2)|v|^2 + H\}dc$$

over all c and m such that

- c is non negative,
- m has density v with respect to c, where $v : Q' \to \mathbb{R}^d$ is a Borelian vector field depending on m and c,
- and, for all admissible ϕ and p,

$$\langle c, \partial_t \phi + p\rangle + \langle m, \nabla_x \phi\rangle = \langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle + \int_Q p(t,x) dt dx. \tag{2.1}$$

The infimum is equal to $+\infty$ if there is no (c, m) that can be considered. Equation (2.1) implies that (c, m) is a generalized flow, but it also contains the boundary condition on $(\mathbb{R}/T\mathbb{Z}) \times \{0, L\} \times \Sigma \times A$, as ϕ in (2.1) is not required to vanish on $(\mathbb{R}/T\mathbb{Z}) \times \{0, L\} \times \Sigma \times A$.

Notations.

- We use the shorter notations

$$\langle c, \partial_t \phi + p\rangle + \langle m, \nabla_x \phi\rangle = \int_{Q'} (\partial_t \phi + p) dc + \int_{Q'} \nabla_x \phi \cdot dm$$

and

$$\langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle = \int_{(\mathbb{R}/T\mathbb{Z})\times\Sigma\times A} \phi(\cdot, L, \cdot, \cdot) d\mu_L$$

- We shall use with the same meaning dc or $c(dt, dx, da)$.
- The notation $c(t, x, da)$ is for the disintegration of c with respect to the projection $(t, x, a) \to (t, x)$; in other words, $\int_A c(t, x, da) = 1$ for almost all t and x (with respect to the Lebesgue measure), and

$$\int_{Q'} f\, dc = \int_Q \left\{ \int_A f c(t, x, da) \right\} dt dx \quad \text{for all} \quad f \in C(Q').$$

For the notion of disintegration of measures, see e.g. Theorem 5.3.1 in [3] or Sect. 10.2 in [11].
- "m has density v with respect to c" means that

$$\int_{Q'} \Phi \cdot dm = \int_{Q'} \Phi \cdot v dc \quad \text{for all } \Phi \in C(Q', \mathbb{R}^d),$$

where $v : Q' \to \mathbb{R}^d$ is assumed to be Borelian and such that $\int |v| dc < \infty$; we use the notation $dm = v\, dc$ or $m = v\, c$.

Dual problem. To study the supremum of

$$\langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle + \int_Q p(t,x) dt dx$$

over all admissible ϕ and p such that

$$\partial_t \phi + p + (1/2)|\nabla_x \phi|^2 \le H \tag{2.2}$$

everywhere on Q'.

Inequality. If c, m satisfy all conditions of the primal problem and ϕ, p all conditions of the dual problem (that is, they can be considered in the inf and sup), the following inequalities hold (where $dm = v\, dc$):

$$\begin{aligned} &\langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle + \int_Q p(t,x) dt dx \\ &= \langle c, \partial_t \phi + p\rangle + \langle m, \nabla_x \phi\rangle \\ &\le \langle c, -(1/2)|\nabla_x \phi|^2 + H\rangle + \langle m, \nabla_x \phi\rangle \\ &= \langle c, -(1/2)|\nabla_x \phi|^2 + v \cdot \nabla_x \phi + H\rangle \le \langle c, (1/2)|v|^2 + H\rangle \end{aligned} \tag{2.3}$$

with equalities if and only if

$$\partial_t \phi + p + (1/2)|\nabla_x \phi|^2 = H \text{ and } v = \nabla_x \phi \tag{2.4}$$

c-almost everywhere.

Before stating the fundamental existence result for the primal problem, let us give a corollary of the previous inequality that is useful to check that a given generalized flow is minimal.

Proposition 2.1. *Let (c, m) be a generalized flow such that the boundary condition (2.1) holds and such that m has the density v with respect to c.*

Let $p : Q \to \mathbb{R}$ and $\phi : Q' \to \mathbb{R}$ be admissible such that

$$\partial_t \phi + p + (1/2)|\nabla_x \phi|^2 \leq H$$

everywhere on Q'.

If (2.4) holds c-almost everywhere, then (c, m) is a minimizer for the primal problem and (ϕ, p) is a maximizer for the dual problem.

The next proposition ensures the existence of (c, m) that realizes the infimum in the primal problem if this infimum is finite. Indeed, the functional $(c, m) \to \int \{(1/2)|v|^2 + H\} dc$ is convex and lower semi-continuous (it is equal to $+\infty$ if c is negative on some Borelian set or if m does not have a density with respect to c). In fact it is equal to the functional $\alpha^* + \beta^*$ appearing below in the proof of the proposition. We choose to work on $\mathbb{R}/T\mathbb{Z}$ to have some compactness available. As A is also compact, so is $Q' = (\mathbb{R}/T\mathbb{Z}) \times D \times A$, and hence the topological dual space of $C(Q')$ is linearly isomorphic to the set of finite signed measures on Q' (see e.g. Thm 7.4.1 in [11]).

Proposition 2.2. *The values of the inf in the primal problem and the sup in the dual problem are either both $+\infty$ or both finite and equal. If the value of the inf is finite, then it is attained and the inf is thus a min.*

Proof. If (c, m) is such that

$$\langle c, \partial_t \phi + p \rangle + \langle m, \nabla_x \phi \rangle = \langle \mu_L, \phi(\cdot, L, \cdot, \cdot) \rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot) \rangle + \int_Q p(t, x) dt dx$$

for all admissible ϕ and p, then the following push-forwards are trivial: $P_{t\sharp}(\mu_L - \mu_0) = 0$ and $P_{a\sharp}(\mu_L - \mu_0) = 0$ (the push-forwards by the projections $(t, x, a) \to P_t(t, x, a) := t$ and $(t, x, a) \to P_a(t, x, a) := a$ respectively). To see it, choose ϕ of the form $f(a) + g(t)$ with f and g continuous, g being C^1 and T-periodic, and $p = -g'$.

First assume that $P_{a\sharp}(\mu_L - \mu_0)$ or $P_{t\sharp}(\mu_L - \mu_0)$ is not trivial. Then the inf in the primal problem is equal to $+\infty$, because there is no (c, m) that can be considered in the primal problem. Moreover we can choose $f \in C(A)$ and $g \in C^1(\mathbb{R}/T\mathbb{Z})$ such that

$$\int \{f(a) + g(t)\} d(\mu_L - \mu_0) > 0.$$

For $\lambda > 0$, we then set $\phi(t, x, a) = \lambda\{f(a) + g(t)\}$ and $p = -\lambda \partial_t g + \min H$ in the dual problem. Letting $\lambda \to +\infty$, this shows that the sup in the dual problem is also $+\infty$.

Secondly, assume that $P_{a\sharp}(\mu_L - \mu_0) = 0$ and $P_{t\sharp}(\mu_L - \mu_0) = 0$. We follow Brenier [8] and Villani [19] (Thm 1.3 and Thm 1.9) by setting, for all $F \in C(Q')$ and $\Phi \in C(Q', \mathbb{R}^d)$,

$$\alpha(F, \Phi) = \begin{cases} 0 & \text{if } F + \frac{1}{2}|\Phi|^2 \leq H \text{ over } Q', \\ +\infty & \text{else.} \end{cases}$$

For all $F \in C(Q')$ and $\Phi \in C(Q', \mathbb{R}^d)$, we also set

$$\beta(F, \Phi) = \langle \mu_L, \phi(\cdot, L, \cdot, \cdot) \rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot) \rangle + \int_Q p(t, x) dt dx$$

if Φ, F are of the form $F = \partial_t\phi + p$ and $\Phi = \partial_x\phi$ for some admissible ϕ and p, else $\beta(F, \Phi) = +\infty$. As $P_{a\sharp}(\mu_L - \mu_0) = 0$ and $P_{t\sharp}(\mu_L - \mu_0) = 0$, the value of $\beta(F, \Phi)$ does not depend on the choice of p and ϕ.

The above dual problem consists in studying the supremum

$$\sup\{-\alpha(F, \Phi) - \beta(-F, -\Phi) : \ F \in C(Q'), \Phi \in C(Q', \mathbb{R}^d)\}.$$

As α is continuous at $\tilde{F} = \min H - 1$ and $\tilde{\Phi} = 0$, and as β is finite at $-\tilde{F}$ and $-\tilde{\Phi}$ (in $-\tilde{F} = \partial_t\tilde{\phi} + \tilde{p}$, choose $\tilde{\phi} = 0$ and $\tilde{p} = -\tilde{F}$), we get that, if the sup is finite, its value equals the following *minimum*:

$$\min\{\alpha^*(c, m) + \beta^*(c, m) : (c, m) \ \text{ is a } \mathbb{R} \times \mathbb{R}^d\text{-valued Borel measure on } Q'\}, \tag{2.5}$$

where α^* and β^* are the convex conjugates of α and β (see e.g. Thm I,11 in [9]). The minimization problem (2.5) is in fact the above primal problem. Indeed

$$\alpha^*(c, m) = \sup\{\langle c, F\rangle + \langle m, \Phi\rangle\},$$

where the sup is taken over the F and Φ such that $F + (1/2)|\Phi|^2 \leq H$. On the other hand,

$$\beta^*(c, m) = \sup\Big\{\langle c, F\rangle + \langle m, \Phi\rangle - \langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle + \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle - \int_Q p(t, x)dtdx\Big\},$$

where the sup is taken over the F, Φ of the form $F = \partial_t\phi + p$ and $\Phi = \partial_x\phi$. □

Brenier's setting [8]. For $d = 2$ or 3, he is interested in incompressible flows inside the spatial domain $[0, 1]^d$ defined on the time interval $[0, T]$ for a given $T > 0$. He also considers a compact metric space A endowed with a Borelian measure (the space of Lagrangian labels).

In this setting, a function ϕ defined on

$$Q' = [0, T] \times [0, 1]^d \times A$$

is said admissible if it is continuous on Q', differentiable with respect to $(t, x) \in [0, T] \times [0, 1]^d$ and such that the derivatives $\partial_t\phi$ and $\nabla_x\phi$ are continuous on Q'.

A generalized flow consists here in a non-negative and finite Borelian measure c on Q' and a vector Borelian measure m on Q' with values in $\mathbb{R}^d$ such that

$$\langle c, \partial_t\phi + p\rangle + \langle m, \nabla_x\phi\rangle = \int_{[0,T]\times[0,1]^d} p \, dtdx$$

for all $p \in C([0, T] \times [0, 1]^d)$ (p does not depend on $a \in A$) and for all admissible ϕ with support in $]0, T[\times[0, 1]^d \times A$.

Given a fixed generalized flow $(\bar{c}, \bar{m})$, the primal problem consists in finding a generalized flow (c, m) that realizes the following infimum:

$$\inf\left\{\int_{Q'} (1/2)|v|^2 dc : v \in L^2(Q', dc)^d, \ dm = v\, dc, \ \langle \bar{c} - c, \partial_t\phi + p\rangle + \langle \overline{m} - m, \nabla_x\phi\rangle = 0 \ \ \forall\phi \text{ admissible } \forall p \in C([0, T] \times [0, 1]^d)\right\}.$$

The given measures $\bar{c}$ and $\bar{m}$ are supposed to satisfy

$$\bar{c}(0, x, da) = \delta(x - i(a)), \ \bar{c}(T, x, da) = \delta(x - h(i(a))),$$

where $i : A \to [0, 1]^d$ is a Borelian bijection, up to negligible sets, that preserves the measures, and $h : [0, 1]^d \to [0, 1]^d$ is a Borelian map that preserves the Lebesgue measure.

Intuitively, the fluid particle labelled by a is located at $x = i(a)$ at time 0 and at $h(x)$ at time T. For $t \in]0, T[$, a particle labelled by a is no more necessarily located at a point, but should rather be considered as a probability measure.

In Brenier's setting, given a continuous function $H : A \to \mathbb{R}$,

$$\int_{Q'} H\,dc = \int_{Q'} H\,d\bar{c} - \int_{Q'} H\,d(\bar{c} - c)$$
$$= \int_{Q'} H\,d\bar{c} - \int_{Q'} \partial_t(tH) d(\bar{c} - c) - \int_{Q'} \nabla_x(tH) \cdot d(\bar{m} - m) = \int_{Q'} H\,d\bar{c}$$

is independent of c. Hence no new information is provided by simply minimizing

$$\int_{Q'} \{(1/2)|v|^2 + H\} dc.$$

There is a similar phenomenon in our setting, namely, given a continuous function $H : (\mathbb{R}/T\mathbb{Z}) \times A \to \mathbb{R}$, define $\overline{H} : A \to \mathbb{R}$ and $\widehat{H} : (\mathbb{R}/T\mathbb{Z}) \times A \to \mathbb{R}$ by

$$\overline{H}(a) = \frac{1}{T} \int_0^T H(t,a)dt \quad \text{and} \quad \widehat{H}(t,a) = \int_0^t \{H(s,a) - \overline{H}(a)\} ds.$$

Then

$$\int_{Q'} H\,dc = \int_{Q'} \partial_t \widehat{H} dc + \int_{Q'} \overline{H}\,dc = \int_{Q'} (\partial_t \widehat{H} + \nabla_x \widehat{H}) dc + \int_{Q'} \overline{H}\,dc = \int_{Q'} \overline{H}\,dc,$$

which shows that no new information is provided by allowing H to depend on t in a periodic way. Analogously, $\int_{Q'} H\,dc$ does not depend on c when H is constant and we can therefore assume that $\int_{Q'} H\,dc = 0$ if convenient.

Brenier's dual problem is given in terms of a continuous function $p : Q \to \mathbb{R}$ and an admissible function $\phi : Q' \to \mathbb{R}$. When $d = 3$, he then proves a striking result about uniqueness and partial regularity of the pressure gradient for optimal solutions of the dual problem. We do not consider the analogous question in our setting.

3. Classical Optimal Solution

We would like to check that, in case the pair of variational problems have a classical optimal solution (c, m, ϕ, p), then it corresponds to a solution to the classical Euler equations (1.1). We shall say that (c, m, ϕ, p) is a classical optimal solution if

1. (c, m) and (ϕ, p) can be considered in the inf and sup in the above primal and dual problems, and the inequalities (2.3) are in fact equalities:
$$\langle c, (1/2)|v|^2 + H \rangle = \langle \mu_L, \phi(\cdot, L, \cdot, \cdot) \rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot) \rangle$$
$$+ \int_Q p(t,x) dt dx < \infty;$$
as a consequence
$$\partial_t \phi + p + (1/2)|\nabla_x \phi|^2 = H \text{ and } v = \nabla_x \phi$$
c-almost everywhere.
2. A is a compact subset of a finite dimensional Euclidean space and there exists a Borelian map $\kappa : Q \to A$ such that $c(t, x, da)$ is the Dirac measure at $\kappa(t, x)$; we denote by $\mathrm{int}(A)$ the interior of A with respect to this finite dimensional Euclidean space.

3. In this case, we can replace $(t, x, a) \to v(t, x, a)$ with
$$(t, x, a) \to v(t, x, \kappa(t, x))$$
without loosing the relationship $dm = v\,dc$; we assume therefore, without of loss of generality, that v does not depend on a.
4. If moreover p is of class C^1, ϕ is of class C^2, κ is of class C^1 and $\kappa(t, x) \in$int(A) for almost all $(t, x) \in Q$, we shall call it "classical and regular".

For a classical and regular optimal solution, we get $\mathrm{div}_x v = 0$ and
$$\partial_t \kappa_j + v \cdot \nabla_x \kappa_j = 0 \tag{3.1}$$
for each component κ_j of κ (that is, κ is preserved by the flow defined by the vector field v). To see it, let $l \geq 1$ and $A \subset \mathbb{R}^l$ have non-empty interior in $\mathbb{R}^l$. For all f compactly supported in $(\mathbb{R}/T\mathbb{Z}) \times \mathrm{int}(D) \times \mathbb{R}^l$ and C^1, we have
$$\int \{\partial_1 f(t, x, \kappa(t, x)) + \partial_2 f(t, x, \kappa(t, x)) \cdot v(t, x)\} dt\, dx = 0.$$
Hence, for all f independent of $a, \int \nabla_x f(t, x) \cdot v(t, x)\, dt\, dx = 0$ and therefore $\mathrm{div}_x v = 0$. For fixed $j \in \{1, \ldots, l\}$ and all $\widetilde{f}$ that depend on t and x, we set $f(t, x, a) = f(t, x, a_j) = a_j \widetilde{f}(t, x)$ and get
$$\begin{aligned} 0 &= \int \{\partial_1 f(t, x, \kappa(t, x)) + \partial_2 f(t, x, \kappa(t, x)) \cdot v(t, x)\} dt\, dx \\ &= \int \{\partial_t f(t, x, \kappa_j(t, x)) - \partial_3 f(t, x, \kappa_j(t, x)) \partial_1 \kappa_j(t, x) \\ &\quad + \nabla_x f(t, x, \kappa_j(t, x)) \cdot v(t, x) - \partial_3 f(t, x, \kappa_j(t, x)) \nabla_2 \kappa_j(t, x) \cdot v(t, x)\} dt\, dx \\ &= -\int \partial_3 f(t, x, \kappa_j(t, x)) \{\partial_1 \kappa_j(t, x) + \nabla_2 \kappa_j(t, x) \cdot v(t, x)\} dt\, dx \\ &= -\int \widetilde{f}(t, x) \{\partial_1 \kappa_j(t, x) + \nabla_2 \kappa_j(t, x) \cdot v(t, x)\} dt\, dx, \end{aligned}$$
which implies $\partial_t \kappa_j + v \cdot \nabla_x \kappa_j = 0$.

If in addition Σ has a smooth boundary, by considering f compactly supported in $(\mathbb{R}/T\mathbb{Z}) \times]0, L[\times\Sigma$ and of class C^1, we get $\int \partial_x f(t, x) \cdot v(t, x)\, dt\, dx = 0$ for all such f. Together with $\mathrm{div}_x v = 0$, the divergence theorem implies that $v(t, x)$ is tangent to the boundary of $]0, L[\times\Sigma$ for all $x \in]0, L[\times\partial\Sigma$.

Proposition 3.1. *For H of class C^1, let (c, m, ϕ, p) be a classical and regular optimal solution. We assume that $A \subset \mathbb{R}^l$, int(A) is understood relatively to $\mathbb{R}^l, a = \kappa(t, x)c$-almost everywhere, v does not depend on a, and that κ and p are of class C^1, and ϕ of class C^2.*

Then
$$\partial_{a_j t} \phi + v \cdot \partial_{a_j} \nabla_x \phi = \partial_{a_j} H \;\text{ at } a = \kappa(t, x) \tag{3.2}$$
for all $j \in \{1, \ldots l\}$ and all $(t, x) \in Q$. Moreover the Euler equation for inviscid and incompressible fluid with constant density holds, namely
$$\partial_t v + (v \cdot \nabla_x) v = -\nabla_x p$$
everywhere on Q.

Proof. As
$$\partial_t \phi + p + (1/2)|\nabla_x \phi|^2 \leq H \text{ on } Q'$$
with equality c-almost everywhere, and as $a = \kappa(t, x) \in \mathrm{int}(A)c$-almost everywhere, we deduce that, for all $(t, x) \in Q$ and for $a = \kappa(t, x)$,
$$\begin{aligned} 0 &= \partial_{a_j} \{\partial_t \phi + p + (1/2)|\nabla_x \phi|^2 - H\} = \partial_{a_j t} \phi + (\nabla_x \phi) \cdot (\partial_{a_j} \nabla_x \phi) - \partial_{a_j} H \\ &= \partial_{a_j t} \phi + v \cdot (\partial_{a_j} \nabla_x \phi) - \partial_{a_j} H \end{aligned}$$

In the same way, $\nabla_x\{\partial_t\phi + p + (1/2)|\nabla_x\phi|^2\}$ vanishes at $a = \kappa(t,x)$ for almost all $(t,x) \in Q$ and thus for all $(t,x) \in Q$. This gives

$$\begin{aligned} 0 &= \partial_t\nabla_x\phi + \nabla_x p + \{(\nabla_x\phi)\cdot\nabla_x\}\nabla_x\phi = \partial_t\nabla_x\phi + \nabla_x p + (v\cdot\nabla_x)\nabla_x\phi \\ &= \partial_t v + \nabla_x p + (v\cdot\nabla_x)v - \sum_{j=1}^{l}(\partial_{a_j}\nabla_x\phi)\{\partial_t + (v\cdot\nabla_x)\}\kappa_j \\ &= \partial_t v + \nabla_x p + (v\cdot\nabla_x)v \end{aligned}$$

(see (3.1)). □

Application: computation of vorticity in $\mathbb{R}^3$. For $j \in \{1,\dots,l\}$, we obtain

$$\begin{aligned} \mathrm{rot}v &= (\partial_{z_1}v_{z_2} - \partial_{z_2}v_{z_1}\ ,\ \partial_{z_2}v_y - \partial_y v_{z_2}\ ,\ \partial_y v_{z_1} - \partial_{z_1}v_y) \\ &= \sum_{j=1}^{l}\Big(\partial_{a_jz_2}\phi\partial_{z_1}\kappa_j - \partial_{a_jz_1}\phi\partial_{z_2}\kappa_j,\ \partial_{a_jy}\phi\partial_{z_2}\kappa_j - \partial_{a_jz_2}\phi\partial_y\kappa_j,\ \partial_{a_jz_1}\phi\partial_y\kappa_j - \partial_{a_jy}\phi\partial_{z_1}\kappa_j\Big) \\ &= \sum_{j=1}^{l}\nabla_x\kappa_j \times \partial_{a_j}\nabla_x\phi. \end{aligned}$$

Moreover, by the first part of the proposition, $\partial_{a_jt}\phi + v\cdot\partial_{a_j}\nabla_x\phi = \partial_{a_j}H$ and therefore

$$\begin{aligned} v\times\mathrm{rot}v &= \sum_{j=1}^{l} v\times(\nabla_x\kappa_j\times\partial_{a_j}\nabla_x\phi) \\ &= \sum_{j=1}^{l}\{\langle v,\partial_{a_j}\nabla_x\phi\rangle\nabla_x\kappa_j - \langle v,\nabla_x\kappa_j\rangle\partial_{a_j}\nabla_x\phi\} \\ &= \sum_{j=1}^{l}\{(\partial_{a_j}H - \partial_{a_jt}\phi)\nabla_x\kappa_j + \partial_t\kappa_j\partial_{a_j}\nabla_x\phi\} \\ &= \nabla_x(H\circ\kappa) + \sum_{j=1}^{l}\{-\partial_{a_jt}\phi\nabla_x\kappa_j + \partial_t\kappa_j\partial_{a_j}\nabla_x\phi\}. \end{aligned}$$

Hence

$$\begin{aligned} \partial_t v + \mathrm{rot}v\times v &= \partial_t\nabla_x\phi - \nabla_x(H\circ\kappa) + \sum_{j=1}^{l}\partial_{a_jt}\phi\nabla_x\kappa_j \\ &= \nabla_x\Big(\partial_1\phi(t,x,\kappa(t,x)) - (H\circ\kappa)\Big). \end{aligned} \tag{3.3}$$

Suppose that ϕ and κ_j are independent of t and that $H\circ\kappa$ is not constant on $\Sigma\times\{0\}$. Then $v\times\mathrm{rot}v = \nabla_x(H\circ\kappa)$ and thus $\mathrm{rot}v$ does not identically vanish.

Stationary flows in $\mathbb{R}^3$. When $d = 3$ Euler's equation (1.1) can be written

$$\partial_t v + \mathrm{rot}v\times v = -\nabla_x\widetilde{H},\ \ \widetilde{H} = (1/2)|v|^2 + p,\ \ \mathrm{div}_x v = 0$$

(see p. 153 in [17]). If the flow is stationary (that is, v and p do not depend on time) and $\nabla\widetilde{H}$ never vanishes, the surfaces $\widetilde{H}$ =constant are filled with stream lines and vortex lines that are transported by the flow. By stream line, it is meant a curve that is everywhere tangent to v and, by vortex line, a curve that is everywhere tangent to rot v. Whereas it is obvious that a stream line is transported by the (stationary) flow, the same result for vortex lines is referred to as the second Helmholtz theorem (see Sect. 25 in [17]). Serrin [17] calls the surfaces $\widetilde{H}$ =constant "Lamb's surfaces". To give $\widetilde{H}$ on the boundary amounts to giving the intersections of the Lamb surfaces with the boundary. For a classical and regular optimal solution (c, m, ϕ, p) such that ϕ, p and κ are independent of time, the function H we have introduced in the variational problem and $\widetilde{H}$ are related by $\widetilde{H}(x) = H(\kappa(x))$ on D (see (2.4) and also (3.3)).

Proposition 3.2. *Under the same hypotheses as in Proposition 3.1, let $T_0 > 0$ and $q \in C^2([0, T_0], D)$ satisfy $q'(t) = v(t, q(t))$ over $[0, T_0]$. Then*

$$\int_0^{T_0} \{(1/2)|\widetilde{q}'(t)|^2 - p(t, \widetilde{q}(t))\}dt \geq \int_0^{T_0} \{(1/2)|q'(t)|^2 - p(t, q(t))\}dt$$

for all $\widetilde{q} \in C^2([0, T_0], D)$ such that $\widetilde{q}(0) = q(0)$ and $\widetilde{q}(T_0) = q(T_0)$.

Proof. Set $a_0 = \kappa(t, q(t)) \in A \subset \mathbb{R}^l$, which does not depend on $t \in [0, T_0]$:

$$\frac{d}{dt}\kappa_j(t, q(t)) = \partial_1\kappa_j(t, q(t)) + \partial_2\kappa_j(t, q(t)) \cdot v(t, q(t)) = 0$$

for $1 \leq j \leq l$, by (3.1). By (2.2) and (2.4), we get

$$\begin{aligned}
&\int_0^{T_0} \{(1/2)|\widetilde{q}'(t)|^2 - p(t, \widetilde{q}(t))\}dt - \int_0^{T_0} \{(1/2)|q'(t)|^2 - p(t, q(t))\}dt \\
&\geq \int_0^{T_0} \{(1/2)|\widetilde{q}'(t)|^2 + (1/2)|\nabla_x\phi(t, \widetilde{q}(t), a_0)|^2 + \partial_1\phi(t, \widetilde{q}(t), a_0) - H(a_0)\}dt \\
&\quad - \int_0^{T_0} \{(1/2)|q'(t)|^2 + (1/2)|\nabla_x\phi(t, q(t), \kappa(t, q(t)))|^2 \\
&\quad + \partial_1\phi(t, q(t), \kappa(t, q(t))) - H(\kappa(t, q(t)))\}dt \\
&= \int_0^{T_0} \{(1/2)|\widetilde{q}'(t)|^2 + (1/2)|\nabla_x\phi(t, \widetilde{q}(t), a_0)|^2 - (1/2)|q'(t)|^2 \\
&\quad -(1/2)|\nabla_x\phi(t, q(t), a_0)|^2\}dt + \int_0^{T_0} \{\partial_1\phi(t, \widetilde{q}(t), a_0) - \partial_1\phi(t, q(t), a_0)\}dt \\
&= \int_0^{T_0} \{(1/2)|\widetilde{q}'(t)|^2 + (1/2)|\nabla_x\phi(t, \widetilde{q}(t), a_0)|^2 \\
&\quad -(1/2)|q'(t)|^2 - (1/2)|\nabla_x\phi(t, q(t), a_0)|^2\}dt \\
&\quad - \int_0^{T_0} \{\nabla_x\phi(t, \widetilde{q}(t), a_0) \cdot \widetilde{q}'(t) - \nabla_x\phi(t, q(t), a_0) \cdot q'(t)\}dt \\
&= \int_0^{T_0} \{(1/2)|\widetilde{q}'(t)|^2 + (1/2)|\nabla_x\phi(t, \widetilde{q}(t), a_0)|^2 - \nabla_x\phi(t, \widetilde{q}(t), a_0) \cdot \widetilde{q}'(t)\}dt \\
&\geq 0
\end{aligned}$$

because $q'(t) = v(t, q(t)) = \nabla_x\phi(t, q(t), \kappa(t, q(t))) = \nabla_x\phi(t, q(t), a_0)$. □

Remark. Such kind of argument to show that an extremal is a minimizer of a variational integral is standard; see e.g. Sect. 2.6 of Chap. 4 in [12].

4. Comparison with the Stream-Vorticity Formulation in Dimension 2

Let us briefly describe a classical approach [5,13,14] to stationary, incompressible and rotational flows in dimension 2 that we shall interpret in terms of generalized flows. Let $\Sigma = [0,1], D = [0,L] \times \Sigma$ and $\psi \in C^2(\operatorname{int} D) \cap C^1(D)$ satisfy

$$\begin{gathered}\Delta\psi = f(\psi) \text{ on } \operatorname{int} D,\\ \psi([0,L]\times\{0\}) = \{0\},\ \psi([0,L]\times\{1\}) = \{1\},\ \partial_z\psi([0,L]\times]0,1[) \subset]0,\infty[,\end{gathered} \tag{4.1}$$

for some given continuous function $f : \mathbb{R} \to \mathbb{R}$. The vector field $v_\psi = (\partial_2\psi, -\partial_1\psi)$ corresponds to the velocity field of a stationary and incompressible flow, the stream lines being the level sets of ψ and the vorticity being given by $-\Delta\psi$ (more precisely, this is the third component of the vorticity, the two first vanishing). The PDE is nothing else than the preservation of vorticity along stream lines. Let H be a primitive of f.

For $A = [0,1]$ and $T > 0$ arbitrarily chosen, we can associate to ψ the divergence-free vector field $v_\psi = (v_{\psi y}, v_{\psi z}) = (\partial_2\psi, -\partial_1\psi)$ and the generalized flow (c_ψ, m_ψ) on $(\mathbb{R}/T\mathbb{Z}) \times D \times A$ defined as follows:

$$\begin{aligned}\langle c_\psi, E\rangle + \langle m_\psi, \Phi\rangle = &\int_{(\mathbb{R}/T\mathbb{Z})\times D} E(t,x,\psi(x))dtdx\\ &+ \int_{(\mathbb{R}/T\mathbb{Z})\times D} \Phi(t,x,\psi(x))\cdot v_\psi\, dtdx\end{aligned} \tag{4.2}$$

for all $E \in C((\mathbb{R}/T\mathbb{Z}) \times D \times A, \mathbb{R})$ and $\Phi \in C((\mathbb{R}/T\mathbb{Z}) \times D \times A, \mathbb{R}^2)$. As easily seen, the set

$$\{(t,x,a) \in (\mathbb{R}/T\mathbb{Z}) \times D \times A : a = \psi(x)\}$$

has full measure for c_ψ. Moreover the ingoing flux at $y = 0$ and the outgoing flux at $y = L$ are given by $v_{\psi y}$ at $y = 0$ and $y = L$ as follows:

$$\begin{aligned}\langle c_\psi, \partial_t\phi + p\rangle + \langle m_\psi, \nabla_x\phi\rangle = &\int_Q p(t,x)dtdx + \int_{(\mathbb{R}/T\mathbb{Z})\times\Sigma} \phi(t,L,z,\psi(L,z))v_{\psi y}(L,z)dtdz\\ &- \int_{(\mathbb{R}/T\mathbb{Z})\times\Sigma} \phi(t,0,z,\psi(0,z))v_{\psi y}(0,z)dtdz\end{aligned} \tag{4.3}$$

for all admissible ϕ and p.

Let us begin with a particular case that can be dealt with quite easily, and without convexity assumption on H (a primitive of f). Let $\Psi : [0,1] \to \mathbb{R}$ be of class C^2 and such that

$$\Psi(0) = 0,\ \Psi(1) = 1,\ \min \Psi' > 0.$$

Then $\Psi^{-1} : [0,1] \to [0,1]$ exists and is of class C^2. Hence $\Psi'' = f \circ \Psi$, where $f := \Psi'' \circ \Psi^{-1}$ is continuous (and can be extended to all $\mathbb{R}$ if wished). For $L > 0$ fixed, we shall set $\psi(y,z) = \Psi(z)$ for all $x = (y,z) \in [0,L] \times [0,1] := D$. Then $\psi \in C^2(D)$ satisfies (4.1) and, fixing $T > 0$ and setting $A = [0,1]$, we can associate to it a generalized flow (c_ψ, m_ψ) as above. As the derivative of $(1/2){\Psi'}^2 - H(\Psi)$ vanishes, $(1/2){\Psi'}^2 - H(\Psi)$ is constant on $[0,1]$. By choosing the right primitive H of f, it can be therefore assumed that

$$(1/2){\Psi'}^2 - H(\Psi) = 0 \text{ on } [0,1]. \tag{4.4}$$

The next result states that (c_ψ, m_ψ) is a minimizer of a relaxed problem among generalized flows (c,m) with the same ingoing flux at $y = 0$ and outgoing flux at $y = L$ as (c_ψ, m_ψ).

Proposition 4.1. *The generalized flow* (c_ψ, m_ψ) *just defined satisfies*

$$\int\limits_{(\mathbb{R}/T\mathbb{Z})\times D\times A} \{(1/2)|v_\psi|^2 + H(a)\}dc_\psi \leq \int\limits_{(\mathbb{R}/T\mathbb{Z})\times D\times A} \{(1/2)|v|^2 + H(a)\}dc$$

for all generalized flows (c, m) *on* $(\mathbb{R}/T\mathbb{Z}) \times D \times A$ *such that*

$$\begin{aligned}\langle c, \partial_t\phi + p\rangle + \langle m, \nabla_x\phi\rangle = \int\limits_Q p(t,x)dtdx + \int\limits_{(\mathbb{R}/T\mathbb{Z})\times\Sigma} \phi(t, L, z, \psi(L, z))v_{\psi y}(L, z)dtdz\\ - \int\limits_{(\mathbb{R}/T\mathbb{Z})\times\Sigma} \phi(t, 0, z, \psi(0, z))v_{\psi y}(0, z)dtdz \qquad (4.5)\end{aligned}$$

for all admissible ϕ *and* p.

Let $p = 0$ *over* $(\mathbb{R}/T\mathbb{Z}) \times D$ *and define* $\phi \in C^1((\mathbb{R}/T\mathbb{Z}) \times D \times A)$ *by* $\phi(t, y, z, a) = y\Psi'(\Psi^{-1}(a))$. *Then* $(c_\psi, m_\psi, \phi, p)$ *is a classical optimal solution, the role of* κ *in the definition of a classical optimal solution being played by* ψ.

Proof. We apply the criterium of Proposition 2.1: let us check that, for $p = 0$ and ϕ defined at the end of the statement,

$$(1/2)|\nabla_x\phi|^2 \leq H(a) \text{ on } (\mathbb{R}/T\mathbb{Z}) \times D \times A$$

with equality if $a = \Psi(z)$ and that $\nabla_x\phi = (\partial_z\psi(x), -\partial_y\psi(x)) = (\Psi'(z), 0)$ if $a = \Psi(z)$.

In fact

$$(1/2)|\nabla_x\phi|^2 = (1/2)|\Psi'(\Psi^{-1}(a))|^2 \overset{(4.4)}{=} H(a)$$

and equality always holds. Moreover, if $a = \Psi(z)$, then

$$\nabla_x\phi(t, x, a)|_{a=\Psi(z)} = (\Psi'(\Psi^{-1}(\Psi(z))), 0) = (\Psi'(z), 0).$$

□

If $H \in C^2(\mathbb{R})$ and H is convex, ψ has the following standard variational characterization: for all $\widetilde{\psi} \in C^2(\mathrm{int}D) \cap C^1(D)$ such that $\widetilde{\psi}|_{\partial D} = \psi|_{\partial D}$,

$$\begin{aligned}&\int\limits_D \{(1/2)|\nabla\widetilde{\psi}|^2 + H(\widetilde{\psi})\}dx\\ &\quad\geq \int\limits_D \{(1/2)|\nabla\psi|^2 + \nabla\psi\cdot(\nabla\widetilde{\psi} - \nabla\psi) + (1/2)|\nabla\widetilde{\psi} - \nabla\psi|^2 + H(\psi)\\ &\qquad+ f(\psi)(\widetilde{\psi} - \psi)\}dx = \int\limits_D \{(1/2)|\nabla\psi|^2 + H(\psi)\}dx + \int\limits_D (1/2)|\nabla\widetilde{\psi} - \nabla\psi|^2dx\\ &\qquad+ \int\limits_D (-\Delta\psi + f(\psi))(\widetilde{\psi} - \psi)dx \geq \int\limits_D \{(1/2)|\nabla\psi|^2 + H(\psi)\}dx\end{aligned}$$

with equality exactly when $\widetilde{\psi} = \psi$ everywhere. In the next proposition, this is interpreted in the framework of Sect. 2, but the convexity assumption on H is replaced by condition (4.6) below.

Proposition 4.2. *Let $f \in C^1(\mathbb{R})$ and $\psi \in C^3(D)$ satisfy (4.1) and $\partial_z \psi > 0$ on D. Remember the definition (4.2) of (c_ψ, m_ψ) and define $g \in C^1(D)$ by*

$$g(x) = \int_0^y \frac{f(\psi(x))}{\partial_2 \psi(s, \zeta_{\psi(x)}(s))} \, ds$$

where $\zeta_a(s) \in \mathbb{R}$ is defined implicitly by $\psi(s, \zeta_a(s)) = a$ for $s \in [0, L]$ and $a \in A$. Then

$$\int_{(\mathbb{R}/T\mathbb{Z}) \times D \times A} \{(1/2)|v_\psi|^2 + H(a)\} dc_\psi < \int_{(\mathbb{R}/T\mathbb{Z}) \times D \times A} \{(1/2)|v|^2 + H(a)\} dc$$

for all generalized flows (c, m) on $(\mathbb{R}/T\mathbb{Z}) \times D \times A$ such that

$$\begin{aligned} &\int_{(\mathbb{R}/T\mathbb{Z}) \times D \times A} (a - \psi(x)) \nabla g(x) \cdot (v - v_\psi(x)) dc \\ &\quad < \int \left\{ \frac{1}{2}|v - v_\psi(x)|^2 + H(a) - H(\psi(x)) - f(\psi(x))(a - \psi(x)) \right\} dc \end{aligned} \tag{4.6}$$

and such that (4.5) holds for all admissible ϕ and p.

If f vanishes at least once on $A = [0, 1]$, if $\min_A f' > 0$ and if

$$(\max_A f')^2 / (\min_A f')$$

is smaller than some positive real number depending on

$$\max_D \{(\partial_z \psi)^{-1} + |\nabla \psi| + |\psi''|\}$$

and on L, then condition (4.6) is fulfilled for all generalized flows $(c, m) \neq (c_\psi, m_\psi)$ satisfying (4.5).

Proof. Observe that $\zeta_a : [0, L] \to [0, 1]$ is such that $\psi^{-1}(\{a\})$ is the graph of ζ_a:

$$z = \zeta_a(y) \Leftrightarrow \psi(y, z) = a,$$

that the map $(y, a) \to \zeta_a(y)$ is of class $C^3(D \times A)$, that $\zeta_{\psi(y,z)}(y) = z$ and that

$$\zeta_a'(y) = -\frac{\partial_1 \psi(y, \zeta_a(y))}{\partial_2 \psi(y, \zeta_a(y))}, \quad \partial_a \zeta_a(y) = \frac{1}{\partial_2 \psi(y, \zeta_a(y))}.$$

We set

$$p = H(\psi) - |\nabla \psi|^2 / 2$$

and define $h \in C^2(D)$ by

$$h(x) = \int_0^y \left. \frac{|\nabla \psi|^2}{\partial_2 \psi} \right|_{(s, \zeta_{\psi(x)}(s))} ds - \int_0^{\zeta_{\psi(x)}(0)} \partial_1 \psi(0, u) du.$$

We get

$$
\begin{aligned}
\nabla h(x) &= (\partial_2\psi(x), -\partial_1\psi(x)) + \frac{\partial_1\psi(x)}{\partial_2\psi(x)}\,\nabla\psi(x) \\
&\quad + \int_0^y \partial_z \left\{ \frac{|\nabla\psi|^2}{\partial_2\psi} \right\} \frac{1}{\partial_2\psi}\bigg|_{(s,\zeta_{\psi(x)}(s))} ds\,\nabla\psi(x) - \frac{\partial_1\psi(0,\zeta_{\psi(x)}(0))}{\partial_2\psi(0,\zeta_{\psi(x)}(0))}\,\nabla\psi(x) \\
&= (\partial_2\psi(x), -\partial_1\psi(x)) + \int_0^y \left\{ \partial_z \left\{ \frac{|\nabla\psi|^2}{\partial_2\psi} \right\} \frac{1}{\partial_2\psi} + \partial_y \left\{ \frac{\partial_1\psi}{\partial_2\psi} \right\} \right. \\
&\quad \left. - \partial_z \left\{ \frac{\partial_1\psi}{\partial_2\psi} \right\} \frac{\partial_1\psi}{\partial_2\psi} \right\}\bigg|_{(s,\zeta_{\psi(x)}(s))} ds\,\nabla\psi(x) \\
&= (\partial_2\psi(x), -\partial_1\psi(x)) + \int_0^y \frac{\Delta\psi}{\partial_2\psi}\bigg|_{(s,\zeta_{\psi(x)}(s))} ds\,\nabla\psi(x) \\
&= (\partial_2\psi(x), -\partial_1\psi(x)) + \int_0^y \frac{f(\psi(x))}{\partial_2\psi(s,\zeta_{\psi(x)}(s))}\, ds\,\nabla\psi(x) \\
&= (\partial_2\psi(x), -\partial_1\psi(x)) + g(x)\nabla\psi(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla g(x) = (f(\psi(x))/\partial_2\psi(x), 0) &+ \int_0^y \frac{f'(\psi(x))}{\partial_2\psi(s,\zeta_{\psi(x)}(s))}\, ds\,\nabla\psi(x) \\
&- f(\psi(x)) \int_0^y \frac{\partial_{zz}^2\psi}{(\partial_z\psi)^3}\bigg|_{(s,\zeta_{\psi(x)}(s))} ds\,\nabla\psi(x).
\end{aligned}
$$

We also define $\phi \in C^1(D \times A, \mathbb{R})$ by

$$
\phi(x,a) = h(x) + (a - \psi(x))g(x),
$$

the gradient of which is given by

$$
\begin{aligned}
\nabla_x\phi(x,a) &= \nabla h(x) - g(x)\nabla\psi(x) + (a-\psi(x))\nabla g(x) \\
&= (\partial_z\psi(x), -\partial_y\psi(x)) + (a-\psi(x))\nabla g(x).
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\int_{(\mathbb{R}/T\mathbb{Z})\times D\times A} \{(1/2)|v_\psi|^2 + H(a)\} dc_\psi = T\int_D \{(1/2)|\nabla\psi|^2 + H(\psi)\} dx \\
&= T\int_D \{H(\psi) - (1/2)|\nabla\psi|^2\} dx + \int_{(\mathbb{R}/T\mathbb{Z})\times D\times A} \nabla_x\phi(x,a)\cdot(\partial_z\psi(x), -\partial_y\psi(x)) dc_\psi \\
&= \int p(x) dc_\psi + \int \nabla_x\phi(x,a)\cdot v_\psi\, dc_\psi = \int p(x) dc + \int \nabla_x\phi(x,a)\cdot v\, dc
\end{aligned}
$$

for all generalized flows (c, m) satisfying (4.5). Hence

$$\int\limits_{(\mathbb{R}/T\mathbb{Z})\times D\times A} \{(1/2)|v_\psi|^2 + H(a)\}dc_\psi$$
$$= T\int\limits_D \{H(\psi) - (1/2)|\nabla\psi|^2\}dx + \int \{(\partial_z\psi(x), -\partial_y\psi(x))\cdot v$$
$$+(a-\psi(x))\nabla g(x)\cdot v\}dc$$
$$= \int \{H(\psi) - (1/2)|\nabla\psi|^2 + v_\psi\cdot v + (a-\psi(x))\nabla g(x)\cdot(v - v_\psi)$$
$$+(a-\psi(x))f(\psi(x))\}dc$$
$$= \int \{H(\psi(x)) + (a-\psi(x))f(\psi(x)) - H(a) - (1/2)|v - v_\psi|^2$$
$$+(a-\psi(x))\nabla g(x)\cdot(v-v_\psi)\}dc + \int \{(1/2)|v|^2 + H(a)\}dc$$
$$< \int \{(1/2)|v|^2 + H(a)\}dc$$

if (c,m) satisfies (4.6).

To show the last statement, we set $Z = \max_D\{(\partial_z\psi)^{-1} + |\nabla\psi| + |\psi''|\}$, and get $\max_A|f| \le \max_A f'$ and

$$|\nabla g(x)| \le \max_A |f|Z + L\max f'Z^2 + L\max|f|Z^5 = \max_A f'Z(1 + LZ + LZ^4).$$

On the other hand

$$H(a) - H(\psi(x)) - f(\psi(x))(a-\psi(x)) \ge \frac{1}{2}(a-\psi(x))^2 \min_A f'.$$

Hence (4.6) holds true if $(\max_A f')^2Z^2(1+LZ+LZ^4)^2 < \min_A f'$, that is, if

$$\min_A f' > 0 \quad\text{and}\quad \frac{(\max_A f')^2}{\min_A f'} < Z^{-2}(1+LZ+LZ^4)^{-2}. \tag{4.7}$$

□

Remark. If f vanishes at least once on $A = [0,1]$ and (4.7) holds, then it can be checked that the functions p and ϕ introduced in the proof are such that $p(x) + (1/2)|\nabla_x\phi(x,a)|^2 \le H(a)$ on $D\times A$. Therefore $(c_\psi, m_\psi, \phi, p)$ is a classical optimal solution, the role of κ being played by ψ.

5. A Variant of the Formulation in Dimension 2

We use the same notations as before, but we are concerned only with the dimension $d = 2$. We write $x = (y,z) \in D = [0,L]\times[0,1] \subset \mathbb{R}^2$ and we shall say that a continuous map $G : D \to \mathbb{R}$ is *admissible* if $\partial_z G(y,z)$ exists for all $(y,z) \in D, \partial_z G$ is continuous over D and $G(y,1) = 0$ for all $y \in [0,L]$. We assume that $A \subset \mathbb{R}$ is compact.

Primal problem. We seek a finite Borelian measure c on Q' and a vector Borelian measure m on Q' with values in $\mathbb{R}^2$ that realize the following infimum:

$$\inf \int\limits_{Q'} \{(1/2)|v|^2 + H\}dc$$

over all c and m such that c is non negative, $dm = vdc$, that satisfy the new additional condition

$$\int\limits_A ac(t,y,z,da) = \int\limits_0^z \left\{\int\limits_A v_y(t,y,s,a)c(t,y,s,da)\right\} ds \tag{5.1}$$

almost everywhere on Q (with respect to the Lebesgue measure, where v_y denotes the first component of v) and such that, for all admissible ϕ and p,

$$\langle c, \partial_t \phi + p\rangle + \langle m, \nabla_x \phi\rangle = \langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle + \int_Q p(t,x) dt dx.$$

Observe that (5.1) amounts to

$$\int_{Q'} a\partial_z G\, dc = -\int_{Q'} G v_y\, dc \tag{5.2}$$

for all admissible G. In (5.2), we can replace $v_y dc$ by dm_y, where m_y denotes the first component of the vector measure m.

Dual problem. To study the supremum of

$$\langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle + \int_Q p(t,x) dt dx$$

over all admissible ϕ, p and G such that

$$\partial_t \phi + p + a\partial_z G + (1/2)(\partial_y \phi + G)^2 + (1/2)(\partial_z \phi)^2 \le H$$

everywhere.

Inequality. If c, m satisfy all conditions of the primal problem and ϕ, p, G all conditions of the dual problem (that is, they can be considered in the inf and sup), the following inequalities hold (where $dm = v\, dc$):

$$\begin{aligned}
&\langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle + \int_Q p(t,x) dt dx \\
&= \langle c, \partial_t \phi + p\rangle + \langle m, \nabla_x \phi\rangle \\
&\le \langle c, -a\partial_z G - (1/2)(\partial_y \phi + G)^2 - (1/2)(\partial_z \phi)^2 + H\rangle + \langle m, \nabla_x \phi\rangle \\
&= \langle c, -a\partial_z G - (1/2)(\partial_y \phi + G)^2 - (1/2)(\partial_z \phi)^2 + v\cdot\nabla_x \phi + H\rangle \\
&\le \langle c, -a\partial_z G + (1/2)|v|^2 - G v_y + H\rangle \overset{(5.2)}{=} \langle c, (1/2)|v|^2 + H\rangle
\end{aligned}$$

with equalities if and only if

$$\partial_t \phi + p + a\partial_z G + (1/2)(\partial_y \phi + G)^2 + (1/2)(\partial_z \phi)^2 = H \text{ and } v = (\partial_y \phi + G, \partial_z \phi) \tag{5.3}$$

c-almost everywhere.

Proposition 5.1. *The values of the inf in the primal problem and the sup in the dual problem are either both $+\infty$ or both finite and equal. If the value of the inf is finite, then it is attained and the inf is thus a min.*

Proof. Let us sketch the proof in the case that

$$P_{a\sharp}(\mu_L - \mu_0) = 0 \;\text{ and }\; P_{t\sharp}(\mu_L - \mu_0) = 0.$$

We set as in the proof of Proposition 2.2, for all $F \in C(Q')$ and $\Phi \in C(Q', \mathbb{R}^2)$,

$$\alpha(F, \Phi) = \begin{cases} 0 & \text{if } F + \frac{1}{2}|\Phi|^2 \le H \text{ over } Q', \\ +\infty & \text{else,} \end{cases}$$

where $\Phi = (\Phi_y, \Phi_z) \in C(Q', \mathbb{R}^2)$. For all $F \in C(Q')$ and $\Phi \in C(Q', \mathbb{R}^2)$, we now set

$$\beta(F, \Phi) = \langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle + \int_Q p(t,x)dtdx$$

if Φ, F are of the form $F = \partial_t\phi + p + a\partial_z G$ and $\Phi = \partial_x\phi + (G, 0)$ for some admissible ϕ, p and G, else $\beta(F, \Phi) = +\infty$. As $P_{a\sharp}(\mu_L - \mu_0) = 0$ and $P_{t\sharp}(\mu_L - \mu_0) = 0$, the value of $\beta(F, \Phi)$ does not depend on the choice of p, G and ϕ. To see it, it suffices to check that

$$\langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle + \int_Q p(t,x)dtdx = 0$$

if $0 = \partial_t\phi + p + a\partial_z G$ and $0 = \nabla_x\phi + (G, 0)$ for admissible ϕ, p, G. Clearly such a ϕ does not depend on $z, \nabla_x\phi$ neither and thus G neither. Hence $0 = \partial_t\phi + p$ and thus $\partial_t\phi$ does not depend on a. We get, for any fixed $a_0 \in A$,

$$\int_{Q'} p\, dt\, dx = -\int_{Q'} \partial_t\phi(t, x, a_0)\, dt\, dx = 0.$$

As $\partial_t\phi$ depends only on t and y, we deduce that

$$\phi(t, y, z, a) = \phi_1(t, y) + \phi_2(y, a) \text{ with } \phi_1(0, y) = 0$$

(we already know that ϕ does not depend on z). As $\partial_y\phi$ only depends on y, it follows that $\phi(t, y, z, a) = \phi_3(t) + \phi_4(a) + \phi_5(y)$. Finally

$$\phi_5'(y) = \partial_y\phi(t, y, 1, a) = -G(y, 1) = 0$$

and thus

$$\phi(t, 0, z, a) = \phi(t, L, z, a) = \phi_3(t) + \phi_4(a) + \text{Const.}$$

Hence $\langle \mu_L, \phi(\cdot, L, \cdot, \cdot)\rangle - \langle \mu_0, \phi(\cdot, 0, \cdot, \cdot)\rangle = 0$.

The above dual problem consists in studying the supremum

$$\sup\{-\alpha(F, \Phi) - \beta(-F, -\Phi) :\ F \in C(Q'), \Phi \in C(Q', \mathbb{R}^2)\}.$$

As α is continuous at $\tilde{F} = \min H - 1$ and $\tilde{\Phi} = 0$, and as β is finite at $-\tilde{F}$ and $-\tilde{\Phi}$ (in $-\tilde{F} = \partial_t\tilde{\phi} + \tilde{p} + a\partial_z\tilde{G}$, choose $\tilde{\phi} = \tilde{G} = 0$ and $\tilde{p} = -\tilde{F}$), we get as previously that, if the sup is finite, its value equals the following *minimum*:

$$\min\{\alpha^*(c, m) + \beta^*(c, m) :\ (c, m) \text{ is a } \mathbb{R} \times \mathbb{R}^2\text{-valued Borel measure on } Q'\}, \tag{5.4}$$

where α^* and β^* are the convex conjugates of α and β. The minimization problem (5.4) is the above primal problem. $\square$

Classical optimal solution. We shall say that (c, m, ϕ, p, G) is a classical optimal solution if

1. (c, m) and (ϕ, p, G) can be considered in the inf and sup in the above primal and dual problems, and the above inequality is in fact an equality.
2. There exists a Borelian map $\kappa : Q \to A \subset \mathbb{R}$ such that $c(t, x, da)$ is the Dirac measure at $\kappa(t, x)$; we denote by $\text{int}(A)$ the interior of A with respect to $\mathbb{R}$.
3. In this case, we can replace $(t, x, a) \to v(t, x, a)$ with

   $$(t, x, a) \to v(t, x, \kappa(t, x))$$

   without loosing the relationship $dm = v\, dc$; we assume therefore, without of loss of generality, that v does not depend on a.
4. If moreover G and $\partial_z G$ are of class C^1, p is of class C^1, ϕ is of class C^2, κ is of class C^1 and $\kappa(t, x) \in\text{int}(A)$ for almost all $(t, x) \in Q$, we shall call it "classical and regular".

For a classical and regular optimal solution, (5.1) gives $\partial_z\kappa = v_y$.

Proposition 5.2. *For H of class C^1, let (c, m, ϕ, p, G) be a classical and regular optimal solution such that κ does not depend on t and v_y never vanishes on $(\mathbb{R}/T\mathbb{Z})\times]0, L[\times]0, 1[$.*

Then the Euler equation for inviscid and incompressible fluid with constant density holds, namely $\partial_t v + (v \cdot \nabla_x)v$ is spatially a gradient:

$$\partial_t v + (v \cdot \nabla_x)v = -\nabla_x(p + \kappa\partial_z G)$$

over Q.

Proof. As

$$\partial_t\phi + p + a\partial_z G + (1/2)|\partial_y\phi + G|^2 + (1/2)|\partial_z\phi|^2 \leq H \text{ on } Q'$$

with equality c-almost everywhere, we deduce that

$$\nabla_x\{\partial_t\phi + p + a\partial_z G + (1/2)|\nabla_x\phi|^2 + G\partial_y\phi + (1/2)G^2\}$$

vanishes at $a = \kappa(t, x)$ for almost all $(t, x) \in Q$ and thus for all $(t, x) \in Q$. This gives

$$\begin{aligned}
0 &= \partial_t\nabla_x\phi + \nabla_x p + \kappa\nabla_x\partial_z G + \{(\nabla_x\phi)\cdot\nabla_x\}\nabla_x\phi + G\partial_y\nabla_x\phi + \partial_y\phi\nabla_x G + G\nabla_x G\\
&= \partial_t\{\nabla_x\phi + (G, 0)\} + \nabla_x p + \kappa\nabla_x\partial_z G + (v\cdot\nabla_x)\nabla_x\phi + v_y\nabla_x G\\
&= \partial_t\{\nabla_x\phi + (G, 0)\} + (v\cdot\nabla_x)\{\nabla_x\phi + (G, 0)\} + \nabla_x p + \kappa\nabla_x\partial_z G\\
&\quad -(v\cdot\nabla_x G, 0) + v_y\nabla_x G\\
&= \partial_t v + (v\cdot\nabla_x)v - (\partial_a\nabla_x\phi)\{\partial_t + (v\cdot\nabla_x)\}\kappa + \nabla_x p + \kappa\nabla_x\partial_z G + (-v_z\partial_z G, v_y\partial_z G)\\
&= \partial_t v + (v\cdot\nabla_x)v + \nabla_x p + \nabla_x(\kappa\partial_z G)
\end{aligned}$$

where we have used (5.1), which gives here $\partial_z\kappa = v_y$, and

$$v\cdot\nabla_x\kappa = \partial_t\kappa + v\cdot\nabla_x\kappa = 0,$$

which then implies $\partial_y\kappa = -v_z$ because v_y is assumed to never vanish on $(\mathbb{R}/T\mathbb{Z})\times]0, L[\times]0, 1[$. □

Let us go back to the classical approach to stationary, incompressible and rotational planar flows seen in Sect. 4. Let $\Sigma = [0, 1], D = [0, L]\times\Sigma$ and $\psi \in C^2(\text{int } D)\cap C^1(D)$ satisfy (4.1) for some given continuous function $f : \mathbb{R} \to \mathbb{R}$. Remember that the vector field $v_\psi = (\partial_2\psi, -\partial_1\psi)$ corresponds to the velocity field of a stationary and incompressible flow, the stream lines being the level sets of ψ and the vorticity being given by $-\Delta\psi$. For $A = [0, 1]$ and $T > 0$ arbitrarily chosen, we can associate to ψ the divergence-free vector field $v_\psi = (v_{\psi y}, v_{\psi z}) = (\partial_2\psi, -\partial_1\psi)$ and the generalized flow (c_ψ, m_ψ) on $(\mathbb{R}/T\mathbb{Z}) \times D \times A$ defined by (4.2). In addition to (4.3), it satisfies (5.1) (because $\partial_z\psi = v_{\psi y}$).

The next result states that (c_ψ, m_ψ) is a minimizer of a relaxed problem among generalized flows (c, m) with the same ingoing flux at $y = 0$ and outgoing flux at $y = L$ as (c_ψ, m_ψ), and that satisfy the additional condition (5.1).

Proposition 5.3. *Let $f \in C^1(\mathbb{R}), \min f'([0, 1]) \geq 0$ and $\psi \in C^3(D)$ satisfy (4.1) on D. We define $H(s) = \int_0^s f(\sigma)d\sigma$,*

$$p = H(\psi) - \psi f(\psi) - |\nabla\psi|^2/2,$$
$$G(y, z) = -\int_z^1 f(\psi(y, s))ds$$

and $\phi \in C^2(D, \mathbb{R})$ (up to a constant) by

$$\nabla\phi(x) = (\partial_z\psi - G, -\partial_y\psi).$$

Then $(c_\psi, m_\psi, \phi, p, G)$ is a classical optimal solution, the role of κ being played by ψ.

Proof. In fact ϕ does not depend on t and a, and is well defined (up to a constant) because

$$\partial_z(\partial_z\psi - G) - \partial_y(-\partial_y\psi) = \Delta\psi - \partial_z G = f(\psi) - f(\psi) = 0.$$

We have to check (5.3) c_ψ-almost everywhere and

$$\partial_t\phi + p + a\partial_z G + (1/2)(\partial_y\phi + G)^2 + (1/2)(\partial_z\phi)^2 \le H$$

everywhere. Clearly (5.3) holds c_ψ-almost everywhere by the definitions of p, G and ϕ. Moreover

$$\begin{aligned}&\partial_t\phi + p + a\partial_z G + (1/2)(\partial_y\phi + G)^2 + (1/2)(\partial_z\phi)^2\\ &\quad= H(\psi) - \psi f(\psi) - |\nabla\psi|^2/2 + af(\psi) + (1/2)(\partial_z\psi)^2 + (1/2)(\partial_y\psi)^2\\ &\quad= H(\psi) - \psi f(\psi) + af(\psi) = H(\psi) + H'(\psi)(a-\psi) \le H(a)\end{aligned}$$

because H is convex on $A = [0,1]$. □

6. A Variational Problem in Dimension 3

In the previous sections, we studied a classical formulation in dimension 2. We now propose an analogous formulation in dimension 3 for steady flows, such that the formulation in terms of generalized flows can be seen as a relaxation of it. The existence of a stationary point for the classical formulation seems unsettled in general.

Let $D = [0,L] \times \Sigma$ be a cylinder whose section Σ is bounded with smooth and connected boundary $\partial\Sigma$. We note $x = (y,z,w)$ with $(z,w) \in \Sigma$.

A theorem by Euler (see [7]) ensures that all divergence-free vector fields $v \in C^\infty(D)$ can be written in the form $v = \nabla f \times \nabla g$ with f and g smooth, at least in a local way where v does not vanish. Such f and g are then preserved by the flow generated by v.

Let $S \subset \mathbb{R}^2$ be diffeomorphic to Σ and assume now that

$$\int_D \{(1/2)|\nabla f \times \nabla g|^2 + H(f,g)\}dx$$

is stationary at $\bar f, \bar g \in C^\infty(D)$ in the space of all smooth functions f et g such that

$$(f(x), g(x)) \in \partial S,\ \forall x \in [0,L] \times \partial\Sigma$$

and

$$(f(x), g(x)) = (\bar f(x), \bar g(x)),\ \forall x \in \{0,L\} \times \Sigma.$$

We also assume that $(\bar f, \bar g)$ sends $\{0\} \times \Sigma$ and $\{L\} \times \Sigma$ in a diffeomorphic way onto S, with preserved orientation, and that $H : \mathbb{R}^2 \to \mathbb{R}$ is smooth.

Under these conditions, $v = \nabla f \times \nabla g$ is divergence free[2] and the component

$$v_y = (\nabla f \times \nabla g) \cdot (1,0,0) = \partial_z f \partial_w g - \partial_z g \partial_w f = \partial_z \bar f \partial_w \bar g - \partial_z \bar g \partial_w \bar f$$

of v is prescribed on $\{0,L\} \times \Sigma$. Observe that the total flux through $\{0\} \times \Sigma$ is equal to the area of $(\bar f, \bar g)(\{0\} \times \Sigma) = S$ (up to sign).

Making variations in f and g, we get

$$\begin{aligned}&-\mathrm{div}(\nabla\bar g \times (\nabla\bar f \times \nabla\bar g)) + \partial_f H(\bar f, \bar g) = 0\\ &\qquad\text{and } -\mathrm{div}((\nabla\bar f \times \nabla\bar g) \times \nabla\bar f)) + \partial_g H(\bar f, \bar g) = 0.\end{aligned} \tag{6.1}$$

[2] The following formulas are useful: $\vec f \times (\vec g \times \vec h) = (\vec f \cdot \vec h)\vec g - (\vec f \cdot \vec g)\vec h$, $\mathrm{rot}(\lambda \vec f) = (\nabla\lambda) \times \vec f + \lambda \mathrm{rot}\vec f$ and $\mathrm{div}(\vec f \times \vec g) = \vec g \cdot \mathrm{rot}\vec f - \vec f \cdot \mathrm{rot}\vec g$. See p. 186 in [7].

More precisely, we consider f and g of the form $\bar{f}+\delta f$ and $\bar{g}+\delta g$, and keep the terms that are linear with respect to δf and δg in $\int_D\{(1/2)|\nabla f\times\nabla g|^2+H(f,g)\}dx$, which yields

$$\begin{aligned}&\int_D \Big\{(\nabla\bar{f}\times\nabla\bar{g})\cdot(\nabla\delta f\times\nabla\bar{g})+(\nabla\bar{f}\times\nabla\bar{g})\cdot(\nabla\bar{f}\times\nabla\delta g)\\
&\qquad+\partial_1 H(\bar{f},\bar{g})\delta f+\partial_2 H(\bar{f},\bar{g})\delta g\Big\}dx\\
&\quad=\int_D \Big\{\nabla\delta f\cdot\{\nabla\bar{g}\times(\nabla\bar{f}\times\nabla\bar{g})\}+\nabla\delta g\cdot\{(\nabla\bar{f}\times\nabla\bar{g})\times\nabla\bar{f}\}\\
&\qquad+\partial_1 H(\bar{f},\bar{g})\delta f+\partial_2 H(\bar{f},\bar{g})\delta g\Big\}dx\\
&\quad\overset{\text{Gauss}}{=}\int_D \Big\{-\operatorname{div}\{\nabla\bar{g}\times(\nabla\bar{f}\times\nabla\bar{g})\}\delta f-\operatorname{div}\{(\nabla\bar{f}\times\nabla\bar{g})\times\nabla\bar{f}\}\delta g\\
&\qquad+\partial_1 H(\bar{f},\bar{g})\delta f+\partial_2 H(\bar{f},\bar{g})\delta g\Big\}dx\end{aligned}$$

if $\delta f=\delta g=0$ on ∂D. The assumption that $(\bar{f},\bar{g})$ is a stationary point means that these integrals vanish for all such $(\delta f,\delta g)$, which gives (6.1).

Let us write ∂S in Cartesian form $\partial S=\{(u,v)\in\mathbb{R}^2:\beta(u,v)=0\}$, where β is a smooth function such that $\nabla\beta\neq 0$ on ∂S. Let us denote by n a smooth field of outward unitary normal vectors to the surface $]0,L[\times\partial\Sigma$. If in the above computation we only require that $\delta f=\delta g=0$ on $\{0,L\}\times\Sigma$, we get the additional boundary term

$$\int_{]0,L[\times\partial\Sigma}(\nabla\bar{f}\times\nabla\bar{g})\cdot\{n\times(\nabla\bar{g}\delta f-\nabla\bar{f}\delta g)\}.$$

Consider variations δf and δg that are also compatible with the constraint that $]0,L[\times\partial\Sigma$ is sent in ∂S, in the sense that

$$\partial_1\beta(\bar{f},\bar{g})\delta f+\partial_2\beta(\bar{f},\bar{g})\delta g=0.$$

In this case, we get that vanishing of this boundary term amounts to vanishing of

$$(\nabla\bar{f}\times\nabla\bar{g})\cdot\{n\times(\partial_1\beta(\bar{f},\bar{g})\nabla\bar{f}+\partial_2\beta(\bar{f},\bar{g})\nabla\bar{g})\}$$

at the boundary, which indeed vanishes since $\nabla_x\{\beta(\bar{f},\bar{g})\}$ is colinear with n.

Equation (6.1) can also be written

$$\nabla\bar{g}\cdot\operatorname{rot}\bar{v}+\partial_f H(\bar{f},\bar{g})=0 \text{ and } -\operatorname{rot}\bar{v}\cdot\nabla\bar{f}+\partial_g H(\bar{f},\bar{g})=0,\ \text{with } \bar{v}=\nabla\bar{f}\times\nabla\bar{g}.$$

It then follows that

$$\begin{aligned}\bar{v}\times\operatorname{rot}\bar{v}&=(\nabla\bar{f}\times\nabla\bar{g})\times\operatorname{rot}\bar{v}=(\nabla\bar{f}\cdot\operatorname{rot}\bar{v})\nabla\bar{g}-(\nabla\bar{g}\cdot\operatorname{rot}\bar{v})\nabla\bar{f}\\
&=\partial_f H(\bar{f},\bar{g})\nabla\bar{f}+\partial_g H(\bar{f},\bar{g})\nabla\bar{g}=\nabla_x H(\bar{f},\bar{g}).\end{aligned}$$

The identity (see e.g. p. 151 in [17])

$$\nabla(\frac{1}{2}|\bar{v}|^2)=\bar{v}\times\operatorname{rot}\bar{v}+(\bar{v}\cdot\nabla)\bar{v}$$

gives

$$(\bar{v}\cdot\nabla)\bar{v}-\nabla(\frac{1}{2}|\bar{v}|^2)+\nabla_x H(\bar{f},\bar{g})=0,$$

which is indeed in the form

$$(\bar{v}\cdot\nabla)\bar{v}+\nabla\bar{p}=0 \text{ with } \bar{p}=-\frac{1}{2}|\bar{v}|^2+H(\bar{f},\bar{g}).$$

$H(\bar{f},\bar{g})$ can be seen as the Bernoulli constant, which is preserved by the flow since $\nabla_x(H(\bar{f},\bar{g}))\cdot\bar{v}=0$.

Let us check that $\bar{v}$ is tangent to the boundary $]0,L[\times\partial\Sigma$. For $x\in]0,L[\times\partial\Sigma$, if $\nabla\bar{f}(x)$ and $\nabla\bar{g}(x)$ are linearly independent, then the unitary normal vector n is a multiple of $\nabla_x\{\beta(\bar{f},\bar{g})\}$ and $\bar{v}\cdot n$ is a multiple of $\bar{v}\cdot\nabla_x\{\beta(\bar{f},\bar{g})\}=(\nabla\bar{f}\times\nabla\bar{g})\cdot\nabla_x\{\beta(\bar{f},\bar{g})\}=0$. If $\nabla\bar{f}(x)$ and $\nabla\bar{g}(x)$ are linearly dependent, then $\bar{v}=0$. In any case, $\bar{v}$ is tangent to the boundary $]0,L[\times\partial\Sigma$. The fact that $(\bar{f},\bar{g})$ is a stationary point has not been used here.

Our adaptation of Brenier's formulation can be seen as a relaxation in which v and (f,g) are the unknowns, v being sought among vector measures and (f,g) becomes $a\in A$, where A is chosen such that $S\subset A\subset\mathbb{R}^2$ (this is similar to Young's relaxation method [21]). When $H(f,g)$ does only depend on f, f alone could be replaced by $a\in A\subset\mathbb{R}$.

7. Conclusion

In Sect. 2, the period T is fixed. The theory can of course be applied to any multiple nT of T ($n\in\mathbb{N}$), but, in some sense, nothing new is obtained in this way. To see it, we follow part of Sect. 6 in [6], where Mather's measures are interpreted in the framework of optimal transportation theory. For $n\in\mathbb{N}$ fixed, let

$$\gamma_n=n^{-1}\inf\int_{Q'_n}\{(1/2)|v|^2+H(a)\}dc,$$

where the infimum is over c and m, like in Sect. 2, and Q'_n is defined like Q', but with nT instead of T. In the next proposition, we assume for simplicity that $\gamma_1<\infty$. The measures μ_0 and μ_L are extended periodically from $(\mathbb{R}/T\mathbb{Z})\times\Sigma\times A$ to $(\mathbb{R}/nT\mathbb{Z})\times\Sigma\times A$.

Proposition 7.1. *The value of γ_n is independent of $n\in\mathbb{N}$. If (c^1,m^1) denotes any minimizer corresponding to γ_1, then, for each integer $n\geq 2$, the vector measure (c^n,m^n) defined as follows is a minimizer corresponding to γ_n:*

$$\begin{aligned}\int_{Q'_n}F\,dc^n+\int_{Q'_n}\Phi\cdot dm^n&=\sum_{i=0}^{n-1}\int_{[0,T]\times D\times A}F(i+t,x,a)dc^1(t,x,a)\\&\quad+\sum_{i=0}^{n-1}\int_{[0,T]\times D\times A}\Phi(i+t,x,a)\cdot dm^1(t,x,a)\end{aligned}$$

for all $F\in C(Q'_n,\mathbb{R})$ and $\Phi\in C(Q'_n,\mathbb{R}^d)$. Reciprocally, for $n\geq 2$, let $(\widetilde{c}^n,\widetilde{m}^n)$ be any minimizer corresponding to γ_n and define the vector measure $(\widehat{c}^1,\widehat{m}^1)$ on Q' by

$$\int_{Q'}F\,d\widehat{c}^1+\int_{Q'}\Phi\cdot d\widehat{m}^1=n^{-1}\int_{Q'_n}F\,d\widetilde{c}^n+n^{-1}\int_{Q'_n}\Phi\cdot d\widetilde{m}^n$$

for all $F\in C(Q',\mathbb{R})\subset C(Q'_n,\mathbb{R})$ and $\Phi\in C(Q',\mathbb{R}^d)\subset C(Q'_n,\mathbb{R}^d)$. Then $(\widehat{c}^1,\widehat{m}^1)$ is a minimizer corresponding to γ_1.

Proof. Let (c^1,m^1) and (c^n,m^n) be as the statement. We denote the density of m^1 relative to c^1 by v^1, that is, $dm^1=v^1dc^1$. For each $n\geq 2$, we get $dm^n=v^ndc^n$, where

$$v^n(i+t,x,a)=v^1(t,x,a),\qquad i=0,\dots,n-1,\quad t\in[0,T[,$$

and

$$\gamma_n\leq n^{-1}\int_{Q'_n}\{1/2|v^n|^2+H(a)\}dc^n=\int_{Q'}\{1/2|v^1|^2+H(a)\}dc^1=\gamma_1<\infty$$

Reciprocally, for $n \geq 2$, let $(\widetilde{c}^n, \widetilde{m}^n)$ be any minimizer corresponding to γ_n. For each $j = 0, \ldots, n-1$, let $(\widetilde{c}^{n,j}, \widetilde{m}^{n,j})$ be its time translation by jT, that is,

$$\int\limits_{Q'_n} F d\,\widetilde{c}^{n,j} + \int\limits_{Q'_n} \Phi \cdot d\,\widetilde{m}^{n,j} = \int\limits_{Q'_n} F(t - jT, x, a) d\,\widetilde{c}^n + \int\limits_{Q'_n} \Phi(t - jT, x, a) \cdot d\,\widetilde{m}^n$$

for all $F \in C(Q'_n, \mathbb{R})$ and $\Phi \in C(Q'_n, \mathbb{R}^d)$. By convexity,

$$(\widehat{c}^n, \widehat{m}^n) = \left(n^{-1} \sum_{j=0}^{n-1} \widetilde{c}^{n,j}, n^{-1} \sum_{j=0}^{n-1} \widetilde{m}^{n,j} \right)$$

is also a minimizer corresponding to γ_n. Define the vector measure $(\widehat{c}^1, \widehat{m}^1)$ on Q' by

$$\begin{aligned} \int\limits_{Q'} F d\,\widehat{c}^1 + \int\limits_{Q'} \Phi \cdot d\,\widehat{m}^1 &= n^{-1} \int\limits_{Q'_n} F d\,\widehat{c}^{\,n} + n^{-1} \int\limits_{Q'_n} \Phi \cdot d\,\widehat{m}^n \\ &= n^{-1} \int\limits_{Q'_n} F d\,\widetilde{c}^n + n^{-1} \int\limits_{Q'_n} \Phi \cdot d\,\widetilde{m}^n \end{aligned}$$

for all $F \in C(Q', \mathbb{R}) \subset C(Q'_n, \mathbb{R})$ and $\Phi \in C(Q', \mathbb{R}^d) \subset C(Q'_n, \mathbb{R}^d)$. If we denote the corresponding vector densities by $\widehat{v}^n$ and $\widehat{v}^1$, that is, $d\,\widehat{m}^n = \widehat{v}^n d\,\widehat{c}^n$ and $d\,\widehat{m}^1 = \widehat{v}^1 d\,\widehat{c}^1$, then

$$\gamma_1 \leq \int\limits_{Q'} \{1/2 |\widehat{v}^1|^2 + H(a)\} d\,\widehat{c}^1 = n^{-1} \int\limits_{Q'_n} \{1/2 |\widehat{v}^n|^2 + H(a)\} d\,\widehat{c}^{\,n} = \gamma_n$$

because $\widehat{v}^n(t + jT, x, a) = \widehat{v}^1(t, x, a) \widehat{c}^n$-almost everywhere on $[0, T[\times D \times A$ for all $j = 0, \ldots, n-1$. □

Observe however that if $(\widetilde{c}^n, \widetilde{m}^n)$ corresponds to a classical optimal solution, the statement does not ensure that $(\widehat{c}^1, \widehat{m}^1)$ is classical too. In case no optimal solution for the period T were classical, it could be therefore worth working with a larger period nT and even developing a functional setting that allows one to study the limit $n \to \infty$. This would give some insight into the relationship between generalized optimal solutions and the chaotic behavior of the Euler equations (1.1) seen as a dynamical system, by analogy with Mather's measures reflecting chaotic dynamics in periodic Lagrangian systems. Thus it is desirable to set the problem for more general domains, boundary conditions and dependence in time (not necessarily periodic).

In [15], relabeling of particles is discussed in the framework of classical Hamiltonian fluid mechanics and its relationship to preserved quantities is analyzed with the help of both Lagrangian and Eulerian variables. It seems natural to try to merge this formalism with ours. See also [16]. The relationship with the classical two-dimensional stream-vorticity formulation is only partially understood. In the three-dimensional case, it may be worth investigating the classical three-dimensional variational formulation of Sect. 6 by relying on the present relaxed variational approach. Moreover the paper by Ambrosio-Figalli [2] is an extension of the work by Brenier that should be adapted to the present setting. Finally approximation results in the spirit of the work by Shnirelman [18] should be investigated too.

Acknowledgments. At an early stage I had encouraging discussions with E. Séré, Y. Brenier and P. Bernard. In particular I spent one week in Nice in February 2007 and one week in Paris-Dauphine in January 2008. This work was supported by the Swiss National Science Foundation.

References

[1] Aebischer, B., Borer, M., Kälin, M., Leuenberger, Ch., Reimann, H.M.: Symplectic Geometry. Birkhäuser, Boston (1994)

[2] Ambrosio, L., Figalli, A.: Geodesics in the space of measure-preserving maps and plans. Arch. Ration. Mech. Anal. **194**, 421–462 (2009)
[3] Ambrosio, L., Gigli, N., Savaré, G.: Gradient flows in metric spaces and in the Wasserstein space of probability measures. In: Lectures in Mathematics. ETH Zurich, Birkhaüser (2005)
[4] Arnold, V.I., Khesin, B.: Topological Methods in Hydrodynamics. Springer, Berlin (1998)
[5] Benjamin, T.B.: A unified theory of conjugate flows. Phil. Trans. R. Soc. **269**, 587–643 (1971)
[6] Bernard, P., Buffoni, B.: Optimal mass transportation and Mather theory. J. Eur. Math. Soc. **9**, 85–121 (2007)
[7] Brand, L.: Vector and Tensor Analysis. Wiley, New York (1948)
[8] Brenier, Y.: Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. Commun. Pure Appl. Math. **52**, 411–452 (1999)
[9] Brezis, H.: Analyse fonctionnelle. Théorie et applications. Masson, Paris (1983)
[10] Constantin, A., Ivanov, R.I., Prodanov, E.M.: Nearly-Hamiltonian structure for water waves with constant vorticity. J. Math. Fluid Mech. **10**, 224–237 (2008)
[11] Dudley, R.M.: Real Analysis and Probability. Cambridge University Press, Cambridge (2002)
[12] Giaquinta, M., Hildebrandt, S.: Calculus of Variations I: The Lagrangian Formalism. Springer, Berlin (1996)
[13] Lamb, H.: Hydrodynamics. Cambridge University Press, Cambridge (1932)
[14] Marchioro, C., Pulvirenti, M.: Mathematical theory of incompressible nonviscous fluids. In: Applied Mathematical Sciences, vol. 96. Springer, New York (1994)
[15] Salmon, R.: Hamiltonian fluid mechanics. Ann. Rev. Fluid Mech. **20**, 225–256 (1988)
[16] Seliger, R.L., Whitman, G.B.: Variational principles in continuum mechanics. Proc. R. Soc. Lond. A **305**, 1–25 (1968)
[17] Serrin, J.: Mathematical principles of classical fluid mechanics. Handbuch der Physic **8**, 148 (1959)
[18] Shnirelman, A.I.: Generalized fluid flows, their approximation and applications. Geom. Funct. Anal. **4**, 586–620 (1994)
[19] Villani, C.: Topics in optimal transportation. In: Graduate Studies in Mathematics, vol. 58. American Mathematical Society, Providence (2003)
[20] Wahlén, E.: A Hamiltonian formulation of water waves with constant vorticity. Lett. Math. Phys. **79**, 303–315 (2007)
[21] Young, L.C.: Lectures on the Calculus of Variations and Optimal Control Theory. Saunders, Philadelphia (1969)

B. Buffoni
Section de mathématiques (MATHAA/CAA)
École Polytechnique Fédérale-Lausanne
Station 8
1015 Lausanne
Switzerland
e-mail: boris.buffoni@epfl.ch

(accepted: August 13, 2011; published online: October 27, 2011)